Final Exam: Electricity and Magnetism 322
December 11, 2002

Open book/notes. Point values are given with each question. Total exam is worth 180 points but I will mark it out of 150 with maximum mark being 100%.

1. You can answer the following questions with a few sentences and possibly sketches or equations. Focus on "key words" rather than long explanations.

   (a) What are the boundary conditions on $\vec{E}$ and $\vec{D}$ implied by $\nabla \cdot \vec{D} = \rho_f$ and $\nabla \times \vec{E} = 0$? Give sketches of the "tricks" used to derive these two boundary conditions. (15)

   (b) How do you solve an electrostatic problem where $\rho(\vec{x})$ is given, based on Green's solution when there are no conductors or dielectrics present? Why is the problem so much more difficult when conductors and dielectrics are present? (15)

   (c) State whether the following quantities are scalar, vector, or second-rank tensor: volume charge density, current density, dipole moment, quadrupole moment, electric field, electric potential, and the dot product of two vectors. (15)

   (d) A conducting ellipsoid is held at a potential of 10 V. The interior of the ellipsoid is hollow and contains no dielectric and no free charges. Appeal to the uniqueness theorem to give $V(\vec{x})$ and $\vec{E}(\vec{x})$ inside the ellipsoid. (15)

2. Consider an infinitely long square pipe whose central axis is coincident with the z-axis with sides at $x = \pm a/2$ and $y = \pm a/2$. The conducting sides at $x = \pm a/2$ are grounded and the upper side at $y = a/2$ has a potential of $V_0/2$ and the lower side at $y = -a/2$ has a potential of $-V_0/2$.

   (a) Based on the symmetry of the problem what do you conclude about the dependence of V on x, y, and z. (even, odd, constant) inside of the pipe. (5)

   (b) Consider solving Laplace's equation by separation of variables. Follow the arguments that lead you to the conclusion (7)

$$V(x,y,z) = \sum_{n=0}^{\infty} C_n \cos[(2n+1)\frac{\pi x}{a}] \sinh[\frac{(2n+1)\pi y}{a}]. \tag{1}$$

   (c) Calculate the integral

$$\int_{-a}^{a} \cos\left(\frac{m\pi x}{a}\right) \cos\left((2n+1)\frac{\pi x}{a}\right) dx \tag{2}$$

   where $n = 0, 1, 2, 3, ...$ and $m = 1, 2, 3, ...$ (Even if you can't calculate it look up the answer in the book for partial credit and use this result later in the problem.) (10)

   (d) The boundary condition at the upper plate when extended to the regions outside the pipe is

$$V_{\text{bound}}(x, y = \frac{a}{2}) = \begin{cases} V_0/2 & \text{for } -\frac{a}{2} \le x \le \frac{a}{2} \\ -V_0/2 & \text{for } -a \le x < -\frac{a}{2} \text{ or } \frac{a}{2} < x \le a \end{cases} \tag{3}$$

As discussed in class this choice outside the pipe ensures that the function is balanced about $y = 0$, can be expressed as sum of cosine functions, and still matches the physical boundaries inside the pipe. The proper integral for the LHS of this problem is then

$$\int_{-a}^{a} V_{\text{bound}}(x) \cos\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0 & \text{for } m \text{ even} \\ \frac{2aV_0}{m\pi}(-1)^{\frac{m-1}{2}} & \text{for } m \text{ odd} \end{cases} \tag{4}$$

Now determine the values of C_n and give the solution of the problem. (6)
(partial answer:

$$C_0 = \frac{2V_0}{\pi} \frac{1}{\sinh(\pi/2)} \tag{5}$$

which you can use for the next part)

(e) Sketch the boundary condition as a function of x (V_{bound}) at $y = a/2$. Now sketch the *approximation* to the solution over this region of the upper plate if you kept only the $n = 0$ term. (Examples of this type of sketch appear in your text and I gave examples in class.) Keep the appropriate vertical scale between the two curves. (7)

3. Consider a spherical capacitor with inner shell of radius a and outer shell of radius $3a$. The inner shell is held at a potential of V_0 and the outer shell is grounded.

   (a) What is the general solution to Laplace's equation that has the appropriate symmetry for this problem? (5)

   (b) Match this solution to the boundary conditions at $r = a$ and $r = 3a$ to give $V(\vec{x})$ in the region between the spheres in terms of r, V_0, and a. (5)

   (c) The electric field for this problem is

   $$\vec{E}(\vec{x}) = \frac{3aV_0}{2r^2}\hat{r} \tag{6}$$

   for r between the two shells. The electric field is zero in the other two regions. Give σ the surface charge density on the inner and outer shells by using your knowledge of the boundary conditions for $\vec{E}$. (5)

   (d) Calculate Q the total charge on the inner shell based on the above σ (answer: $Q = 6\pi\epsilon_0 V_0 a$) (5)

   (e) State Gauss' Law in integral form. (5)

   (f) Verify a special case of Gauss' Law by integrating $\vec{E}$ over a sphere of radius $2a$ centred on the origin. (5)

   (g) Calculate the capacitance in terms of a and ϵ_0 based on your results for Q and V. Show that it is consistent with the result for a general spherical capacitor. (5)

4. This question is based on a similar geometry to the previous problem but uses a different approach. Consider a spherical capacitor with inner shell of radius a and an outer shell of radius $3a$. The inner shell has a charge of $Q = 6\pi\epsilon_0 V_0 a$. This is a free charge and V_0 represents the potential difference between the shells *with no dielectric present.* The electrical connection between the shells is severed so that the charge Q must stay on the inner sphere.

   The entire space between the spheres is now filled with an isotropic linear dielectric with $\kappa = 2$.

(a) A general question (no math required): is V_0 still the potential difference between the plates? Given Poisson's equation that relates V and ρ what does this imply about ρ before and after the dielectric is in place. Is the change in ρ due to free or bound charges? (7)

(b) Calculate $\vec{D}$ between the spheres using the symmetry of the problem and the Gaussian surface trick. Calculate the electric field in this region. (8)

(c) Calculate the polarization $\vec{P}$ and the bound charge density ρ_b in the bulk of the dielectric. (5)

(d) Calculate the bound surface charge density σ_b on the inner conductor (there are several ways to do this but if you use the method involving the normal vector $\hat{n}$ remember that it points *away* from the dielectric). (5)

(e) Calculate the path integral of the electric field from inner conductor to the outer conductor to find the potential difference between the spheres. (5)

5. (a) State the continuity equation. What does it imply about electric charge? (5)

(b) Suppose that the current density is in the z-direction but varies with position and is equal to

$$\vec{J}(x, y, z) = (3 + 2z)\hat{k} \tag{7}$$

with $\vec{J}$ in units of A/m^2 and x and z in units of metres. Calculate the current out of the unit cube either directly from $\vec{J}$ or using the continuity equation. (10)

(c) If the conductivity of the medium is 10 Seimens per metre what is the electric field at $z = 0.5$ m. (5)