Final Exam: Electricity and Magnetism 322
December 4, 2003

Open book with one two-sided formula sheet. Point values are given with each question. Total exam is worth 155 points but I will mark it out of 140 with maximum mark being 100%.

1. Suppose we have a dipole $\vec{p} = qd\hat{k}$ that sits at the origin where d is small compared to the other length scales.

   (a) Give the electric potential as a function of r and θ. (5)

   (b) Does this constitute a separable solution of Laplace's equation in spherical coordinates? State $R(r)$ and $\Theta(\theta)$ (sorry, can't find a happy face). What is the other $R(r)$ that pairs with this Θ? (8)

   (c) Sketch the equipotentials and the electric field lines labelling the $V = 0$ equipotential. (7)

2. Consider an infinitely long square pipe whose central axis is coincident with the z-axis with sides at $x = \pm a/2$ and $y = \pm a/2$. The conducting sides at $x = \pm a/2$ are grounded and the upper side at $y = a/2$ has a potential of $V_0/2$ and the lower side at $y = -a/2$ has a potential of $-V_0/2$.

   (a) Based on the symmetry of the problem what do you conclude about the dependence of V on x, y, and z. (even, odd, constant) inside of the pipe. (5)

   (b) Show that

   $$V(x, y, z) = \sum_{n=0}^{\infty} C_n \cos[(2n+1)\frac{\pi x}{a}] \sinh[\frac{(2n+1)\pi y}{a}]. \tag{1}$$

   satisfies Laplace's equation and the symmetry considerations in part (a). Why have I choosen this particular form for "k"? (k is the coefficient of x or y in the trigonometric or hyperbolic function). Demonstrate mathematically what this means. (10)

   The values of C_n are determined using Fourier series as we discussed in class. The first coefficient is

   $$C_0 = \frac{2V_0}{\pi} \frac{1}{\sinh(\pi/2)} \tag{2}$$

   (c) Sketch the boundary condition as a function of x at $y = a/2$ between $x = -\frac{a}{2}$ and $x = \frac{a}{2}$. Now sketch the *approximation* to the solution over this region of the upper plate if you kept only the $n = 0$ term. (I gave an example in class and I am ignoring the "extended" parts here.) Keep the appropriate vertical scale between the two curves. (10)

   (d) Determine the approximate value of $\vec{E}$ at the upper plate by just keeping the C_0 term for the solution of $V(\vec{x})$. What is the charge density σ on the top plate implied by this $\vec{E}$. (Rembember the normal vector points out from the conductor so it is $-\hat{\jmath}$.) What would you expect for the value of $\vec{E}_{\tan}$ at the plate to be for the full solution? (10)

3. (a) State Gauss' Law in integral and differential form. (5)

(b) According to Coulomb's Law the electric field of a point charge at the origin is

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{3}$$

Show that this is consistent with Gauss' Law in integral form if we choose a sphere of radius r centred on the origin as our Gaussian surface. (10)

4. (a) What is the scalar potential of a point charge q at the origin? Give the answer in both spherical polar coordinates and Cartesian coordinates. (5)

   (b) Calculate the electric field in spherical polar coordinates. (5)

   (c) What must $\nabla \times (-\nabla V)$ be equal to? Calculate it to demonstrate this fact for this particular choice of V. (7)

   (d) We know that $-\nabla^2 V = 0$ for $r \neq 0$. Using the Dirac delta function and your knowledge of the Green's function state what $-\nabla^2(\frac{1}{r})$ (or equivalently $-\nabla^2(\frac{1}{|\vec{x}|})$) is for all points, including the origin. (8)

   (e) Suppose instead that

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{qa}{r^2} \tag{4}$$

   where a is some constant with dimensions of distance. What is the charge density ρ for $r \neq 0$? (10)

5. What is the appropriate size and location of an image charge for a grounded conducting sphere of radius a centred on the origin if the real charge is q and is located at $(0, 0, z_0)$? Give the expression for the electric potential in the region outside the sphere (your choice of coordinates). Demonstrate that the potential at the two points $(0, 0, a)$ and $(0, 0, -a)$ (Cartesian) or $r = a, \theta = 0$ and $r = a, \theta = \pi$ (spherical) is zero. (20)

6. Consider a parallel plate capacitor with plate area A and plate separation d. The plate separation is much smaller than the linear dimension of the plate. The lower plate has a charge (this is a free charge) of Q and sits in the $x - y$ plane and the top plate has a charge of $-Q$. The space between the plates is filled with a dielectric with $\kappa = 2$. The displacement field has the form $\vec{D}(\vec{x}) = D(z)\hat{k}$ and is zero in region $z < 0$ and $z > d$.

   (a) Use $\oint \vec{D} \cdot d\vec{A} = Q_{f,\text{enc}}$ to find $\vec{D}$ between the plates in terms of Q, A, and d. Using a Gaussian pillbox with the lower surface at $z < 0$ and upper surface at z is most useful. Explain why some parts of the pillbox don't contribute to the flux integral. (10)

   (b) Use the constitutive relation to find $\vec{E}$. (5)

   (c) What is the general solution to Laplace's equation that has the appropriate symmetry for this problem? If we are going to use a solution to Laplace's equation what is the value of ρ between the plates? (7)

   (d) What is potential difference between the plates (a couple of ways to do this)? What is the capacitance? (8)