Final Exam: Electricity and Magnetism 322
December 8, 2005

Closed book with one two-sided formula sheet and photocopy of front and back covers. I can give you unit vector relations from the book on request. Point values are given with each question. Be careful with your vectors and notation. Unless otherwise indicated state the assumptions you are making. Total exam is worth 149 points but I will mark it out of 140 with maximum mark being 100%.

(40)

1. Do any 4 of the following 5 questions. You can answer with a few sentences and possibly sketches or equations. Focus on "key words" rather than long explanations.

   (a) What are the boundary conditions on $\vec{E}$ and $\vec{D}$ implied by $\nabla \cdot \vec{D} = \rho_f$ and $\nabla \times \vec{E} = 0$? Give sketches of the "tricks" used to derive these two boundary conditions. (no need to give the derivation) (10)

   (b) Explain how using an image charge helps when solving problems with external charges and conductors? Use the example of a charge and an infinite grounded conducting plane for the basis of your reasons if you wish. (10)

   (c) How do you solve an electrostatic problem where $\rho(\vec{x'})$ is given, based on Green's solution when there are no conductors or dielectrics present? Why is the problem so much more difficult when conductors and dielectrics are present? (10)

   (d) Is Coulomb's law physically equivalent to Gauss's Law when we consider problems in electrostatics? Give a specific example to support your conclusion (don't need full mathematical details). (10)

   (e) The recursion relation for a Legendre polynomial of order ℓ is

   $$\frac{C_{n+2}}{C_n} = \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)} \tag{1}$$

   and we must also have $P_\ell(u = 1) = 1$. Show how these two relationships are satisfied for $P_2(u) = \frac{3u^2-1}{2}$, identifying what the various terms are. (10)

(25)

2. Use the integral form of Coulomb's Law

   $$\vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x'}) \frac{(\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3} \tag{2}$$

   to set up the integral for the electric field of an infinite plane of surface charge density σ lying in the $z = 0$ plane. Please include a diagram. (Remember that you need to change $d^3x'\rho$ to a form appropriate for a surface charge.) Based on symmetry (or your previous experience with this problem) what form does $\vec{E}$ take (i.e. choose the best coordinate system, eliminate variables, and components). (25)

(29)

3. Consider an infinitely long square pipe whose central axis is coincident with the z-axis with sides at $x = \pm a/2$ and $y = \pm a/2$. The conducting sides at $x = \pm a/2$ are grounded and the upper side at $y = a/2$ has a potential of $V_0/2$ and the lower side at $y = -a/2$ has a potential of $-V_0/2$.

   (a) Based on the symmetry of the problem what do you conclude about the dependence of V on x, y, and z. (even, odd, constant) inside of the pipe. (3)

(b) Show that

$$V(x,y,z) = \sum_{n=0}^{\infty} C_n \cos[(2n+1)\frac{\pi x}{a}] \sinh[\frac{(2n+1)\pi y}{a}]. \tag{3}$$

satisfies Laplace's equation and the symmetry considerations in part (a). (6)

(c) Why have I choosen this particular form for "k"? (k is the coefficient of x or y in the trigonometric or hyperbolic function). Demonstrate mathematically what this means. (6)

The values of C_n are determined using Fourier series as we discussed in class. The first coefficient is

$$C_0 = \frac{2V_0}{\pi}\frac{1}{\sinh(\pi/2)} \tag{4}$$

(c) Determine the approximate value of $\vec{E}$ at the upper plate by just keeping the C_0 term for the solution of $V(\vec{x})$. What is the charge density σ on the top plate implied by this $\vec{E}$. (Rembember the normal vector points out from the conductor so it is $-\hat{\jmath}$.) What would you expect for the value of $\vec{E}_{\tan}$ at the plate to be for the full solution? (10)

(d) Where is the approximate solution most accurate? Least accurate? (the answer is based on a class discussion and some plots that I showed you) (4)

(23) 4. There are two point charges of q sitting on the z-axis at $z = \pm d$. Consider using a multipole expansion to approximate $V(r,\theta,\phi)$.

(a) Give the exact solution for $V(r,\theta,\phi)$. (Hint: if you are using Law of Cosines $\cos(\pi - \theta) = -\cos\theta$) (7)

(b) Calculate Q_T, $\vec{p}$, and $\mathcal{Q}_2$. State your answers in terms of Cartesian unit vectors or as matricies. Have another look at $\mathcal{Q}_2$ and make sure all of your factors are there! (6).

(c) What is $\hat{r}$ in terms of r, θ, ϕ, and Cartesian unit vectors? (3)

(d) Give the approximate solution for V up to order $1/r^3$. (7)

(32) 5. Consider a sphere of radius a centred on the origin. The potential inside and outside the sphere is described as follows

$$V_{\text{inside}}(r,\theta,\phi) = Ar\cos\theta \tag{5}$$

$$V_{\text{outside}}(r,\theta,\phi) = Aa^3\frac{\cos\theta}{r^2} \tag{6}$$

where A is a constant.

(a) You know that both of these forms are separable solutions to Laplace's equation. What does that say about ρ when you are not at the surface? (2)

(b) What fields do you get from these potentials? (just in words is fine) Sketch $\vec{E}$ in both regions. (7)

(c) Is the sphere a conductor? Why or why not? (2)

(d) Suppose that the sphere has a uniform polarisation $\vec{P} = 3\epsilon_0 A\hat{k}$. Calculate the bound surface charge and show that the boundary conditions for $\vec{E}$ are satisfied at $r = a$. (Hint: remember that $\hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$.) (9)

(e) There is no free charge in the problem.

i. Does this mean that the dispacement field $\vec{D} = 0$? (2)

ii. Calculate $\vec{D}$ in both regions using $\vec{D} = \epsilon_0\vec{E} + \vec{P}$. (6)

iii. Where is $\nabla \times \vec{D} \neq 0$ (or perhaps, where are the tangential components discontinous)? (2)

iv. Where is $\nabla \times \vec{E} \neq 0$? (2)