Final Exam: Electricity and Magnetism 322
April 16, 2005

Closed book with one two-sided formula sheet and photocopy of front and back covers. Point values are given with each question. Total exam is worth 125 points but I will mark it out of 115 with maximum mark being 100%.

1. You can answer the following questions with a few sentences and possibly sketches or equations. Focus on "key words" rather than long explanations.

   (a) What are some of the keys and points to consider when solving a problem in electrostatics? Try to come up with 3 solid facts. (10)

   (b) What are the boundary conditions on $\vec{E}$ and $\vec{D}$ implied by $\nabla \cdot \vec{D} = \rho_f$ and $\nabla \times \vec{E} = 0$? Give sketches of the "tricks" used to derive these two boundary conditions. (10)

   (c) What is the relationship between force, electric field, and charge? (5)

   (d) State the equation that allows you to express $\vec{E}$ as the gradient of a scalar function. (5)

2. Use the integral form of Coulomb's Law

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho \frac{(\vec{x} - \vec{x'})}{|\vec{x} - \vec{x'}|^3} \tag{1}$$

   to set up the integral for the electric field of an infinite line charge of charge density λ lying along the z-axis. (Remember that you need to change $d^3x'\rho$ to a form appropriate for a line charge.) Based on symmetry (or your previous experience with this problem) what form does $\vec{E}$ take (i.e. choose the best coordinate system, eliminate variables, and components). No need to state your reasons. (20)

3. Consider an infinitely long square pipe whose central axis is coincident with the z-axis with sides at $x = \pm a/2$ and $y = \pm a/2$. The conducting sides at $x = \pm a/2$ are grounded and the upper side at $y = a/2$ has a potential of $V_0/2$ and the lower side at $y = -a/2$ has a potential of $-V_0/2$.

   (a) Based on the symmetry of the problem what do you conclude about the dependence of V on x, y, and z. (even, odd, constant) inside of the pipe. (5)

   (b) Show that

$$V(x, y, z) = \sum_{n=0}^{\infty} C_n \cos[(2n+1)\frac{\pi x}{a}] \sinh[\frac{(2n+1)\pi y}{a}]. \tag{2}$$

   satisfies Laplace's equation and the symmetry considerations in part (a). (5)

   (c) What kind of solution is this? What functions do you use when using a similar approach for spherical polar cooridinates? (Be as specific as possible.) (10)

   The values of C_n are determined using Fourier series as we discussed in class. The first coefficient is

$$C_0 = \frac{2V_0}{\pi} \frac{1}{\sinh(\pi/2)} \tag{3}$$

(c) Sketch the boundary condition as a function of x at $y = a/2$ between $x = -\frac{a}{2}$ and $x = \frac{a}{2}$. (i.e. what V should be along the upper plate) Now sketch the *approximation* to the solution over this region of the upper plate if you kept only the $n = 0$ term. (I gave an example in class and I am ignoring the "extended" parts here.) Keep the appropriate vertical scale between the two curves. (5)

4. (a) State Gauss' Law in integral and differential form. (5)

   (b) According to Coulomb's Law the electric field of a point charge at the origin is

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{4}$$

   Show that this is consistent with Gauss' Law in integral form if we choose a sphere of radius r centred on the origin as our Gaussian surface. (10)

5. The appropriate size and location of the image charge for a grounded conducting sphere of radius a centred on the origin if the real charge is q and is located at $(0, 0, z_0)$ is

$$q' = -\frac{aq}{z_0} \tag{5}$$

$$z_0' = \frac{a^2}{z_0} \tag{6}$$

   Give the expression for the electric potential in the region outside the sphere (your choice of coordinates). Demonstrate that the potential at the two points $(0, 0, a)$ and $(0, 0, -a)$ (Cartesian) or $r = a, \theta = 0$ and $r = a, \theta = \pi$ (spherical) is zero. (15)

6. Consider a parallel plate capacitor with plate area A and plate separation d. The plate separation is much smaller than the linear dimension of the plate. The lower plate has a charge (this is a free charge) of Q and sits in the $x - y$ plane and the top plate has a charge of $-Q$. The space between the plates is filled with a dielectric with $\kappa = 2$. The displacement field has the form $\vec{D}(\vec{x}) = D(z)\hat{k}$ and is zero in region $z < 0$ and $z > d$.

   (a) Use $\oint \vec{D} \cdot d\vec{A} = Q_{f,\text{enc}}$ to find $\vec{D}$ between the plates in terms of Q, A, and d. You may assume the free surface charge distribution is uniform and is $\sigma_f = \pm Q/A$. Using a Gaussian pillbox with the lower surface at $z < 0$ and upper surface at z is most useful. You will find that the area of the flat surface of the pill box cancels out and you have no z dependence. (8)

   (b) Use the constitutive relation to find $\vec{E}$. (5)

   (c) What is potential difference between the plates (a couple of ways to do this)? What is the capacitance? (7)