Solutions for Midterm Exam: Electricity and Magnetism 322
February 16, 2005

I didn't expect you to include all of the explanation that I have here.

1. Consider the vector field

$$\vec{F}(\vec{x}) = \hat{r} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \tag{1}$$

(a) Calculate $\nabla \cdot \vec{F}$ in both spherical polar and Cartesian coordinates. (10)
Answer: consult solutions for Assignment #1.

$$\nabla \cdot \vec{F} = \frac{2}{r} \tag{2}$$

(b) Calculate $\oint \vec{F} \cdot d\vec{A}$ for a sphere of radius a centred at the origin using the method of your choice. (5)
Answer: Choose the direct method. The vector field is always normal to the surface so $\vec{F} \cdot d\vec{A} = |\vec{F}|\, dA$. The magnitude of the vector field is also constant on the surface with value 1 so

$$\oint \vec{F} \cdot d\vec{A} = \oint dA = 4\pi a^2 \tag{3}$$

The other method would be the volume integral of the divergence. The problem has spherical symmetry so $\int_V d^3x'$ is replaced with $\int_0^a dr\, 4\pi r^2$. Using the answer from part (a)

$$\int \nabla \cdot \vec{F} \cdot d^3x' = 4\pi \int_0^a dr\, 2r = 4\pi a^2 \tag{4}$$

(c) Can we express $\vec{F}$ as the negative gradient of some scalar function $f(\vec{x})$ (i.e. $\vec{F} = -\nabla f$)? What is the relevant property of $\vec{F}$ that determines this? (5)
Answer: Yes, we can. For example suppose that $f(\vec{x}) = -r$ then $\vec{F} = -\nabla f$. The relavent property is that $\nabla \times \vec{F} = 0$. This is easy to demonstrate in spherical polar coordinates. $F_r = 1$, $F_\theta = 0$, and $F_\phi = 0$. The only non-vanishing terms in the curl involve direct partial derivatives of F_r with respect to θ and ϕ. Again consult your solutions for Assignment #1.

(d) From the Helmholtz theorem or your knowledge of Green's solution of Poisson's equation give the answer for the form of $f(\vec{x})$. (don't need to calculate it). In this particular case I don't think the boundary conditions are strictly satisfied but a solution still exists. (5)
Answer: Using Helmholtz theorem allows you to change to the vector function $\vec{F}$ from its derivatives $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ through the potentials $\psi(\vec{x})$ (scalar) and $\vec{A}(\vec{x})$ (vector). In this case you are after the scalar potential since according to the Helmholtz theorem $\vec{F} = -\nabla\psi + \nabla \times \vec{A}$ and we know that $\vec{F}$ is irrotational or curl-free so you only need the first term. f corresponds to ψ.

$$f(\vec{x}) = \int \frac{\nabla \cdot \vec{F}}{4\pi|\vec{x} - \vec{x'}|} d^3x' = \int \frac{2\, d^3x'}{4\pi|\vec{x'}||\vec{x} - \vec{x'}|} \tag{5}$$

Using your knowledge of Green's solution, recognize from Gauss Law that $\rho/\epsilon_0 = 2/r$ so we just plug this into Green's solution for Poisson's equation

$$V(\vec{x}) = \int \frac{\rho/\epsilon_0}{4\pi|\vec{x} - \vec{x'}|} d^3x' = \int \frac{2\, d^3x'}{4\pi|\vec{x'}||\vec{x} - \vec{x'}|} \tag{6}$$

Calculating this integral might be tough... let's try. Suppose that the direction of $\vec{x}$ defines the direction of $\hat{k}'$. (The primes integrate out so we can choose what we like.) There is axial symmetry about this direction so we handle the ϕ' integral by multiplying by 2π.

$$f(\vec{x}) = \int_0^\infty dr' \int_0^\pi d\theta' \frac{r'^2 \sin\theta}{r'\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} \tag{7}$$

This integral will in general diverge (this is what I meant about the boundary conditions) but if you just assume that R_∞ is where you stop the integration then you find $f(\vec{x}) = -r + 2R_\infty$. This gives you the correct $\vec{F}$.

2. Suppose you have an infinitely long charged cylindrical shell with its axis along the z-axis (like the outer sheath of a coaxial cable). The radius of the cylinder is a and its surface charge density is σ.

   (a) What is the form of $\vec{E}$ based on the symmetry of the charge distribution and properties of $\vec{E}$? (Choose the most meaningful coordinate system and eliminate independent variables and vector components.) Give short justifying statements. (7)

   Answer: First: *know the answer!* These are the same arguments used for a long wire. Consider cylindrical polar coordinates (r, ϕ, z). The charged shell has axial symmetry about the $\hat{k}$ direction so we don't expect any ϕ dependence. I used to have a trick for recognizing this. Suppose I throw you a long piece of chalk and ask you to put a scratch at $\phi = 0$..., or a marble and put a mark at the north pole.... Likewise if the cylinder is infinitely long there will be no z dependence. So we are left with $E_r(r)$, $E_\phi(r)$, and $E_z(r)$. If there were an $E_\phi(r) \neq 0$ then we could find a circular path such that $\oint \vec{E}\cdot d\vec{A} \neq 0$ which would violate $\nabla \times \vec{E} = 0$. Hence, $E_\phi = 0$. If $E_z(r) \neq 0$ then there would be a preferred direction along z which would violate the reflection symmetry of the cylinder. So

$$\vec{E}(\vec{x}) = E_r(r)\hat{r} \tag{8}$$

   where we are using cylindrical polar coordinates.

   (b) Use Gauss Law in integral form to calculate $\vec{E}$ in the regions inside and outside the cylinder. (10)

   Answer: Consider the Gaussian surface to a be cylinder of height h and radius r. The electric field is normal to and constant over the curved surface of the Gaussian cylinder and normal to the ends of the Gaussian cylinder. If $r < a$ then no charge is enclosed and

$$\oint_{cylinder} E_r(r)dA = E_r(r)\,2\pi rh = 0 \tag{9}$$

   so $E_r(r) = 0$ for $r < a$.

   For $r > a$ the amount of charge enclosed is $2\pi ah\sigma$ (area of enclosed charge cylinder times the surface charge density) so

$$E_r(r) = \frac{2\pi ah\sigma}{\epsilon_0}\frac{1}{2\pi rh} = \frac{\sigma a}{\epsilon_0 r} \tag{10}$$

   (c) State the boundary conditions for $\vec{E}$ and show that they are satisfied at the cylindrical shell. (8)

Answer: The boundary conditions are

$$E_{2n} - E_{1n} = \frac{\sigma}{\epsilon_0} \tag{11}$$

$$\vec{E}_{2tan} = \vec{E}_{1tan} \tag{12}$$

The normal vector points from region 1 to region 2 so if region 1 is the interior $\hat{n} = \hat{r}$.

$$E_{2n} = \vec{E}_2(a) \cdot \hat{n} = \frac{\sigma a}{\epsilon_0 a}\hat{r} \cdot \hat{r} = \frac{\sigma}{\epsilon_0} \tag{13}$$

$$E_{1n} = \vec{E}_1(a) \cdot \hat{n} = (0)\hat{r} \cdot \hat{r} = 0 \tag{14}$$

$$\vec{E}_{2tan} = \vec{E}_2 - E_{2n}\hat{n} = 0 \tag{15}$$

$$\vec{E}_{1tan} = \vec{E}_1 - E_{1n}\hat{n} = 0 \tag{16}$$

Boundary conditions are obviously satisfied.

3. Consider an array of 4 charges

$$\begin{aligned} \vec{x}_1 &= d\hat{\imath} \\ \vec{x}_2 &= d\hat{\jmath} \\ \vec{x}_3 &= -d\hat{\imath} \\ \vec{x}_4 &= -d\hat{\jmath}. \end{aligned}$$

Charges 1 and 3 are q and charges 2 and 4 are $-q$.

(a) Calculate Q_T, $\vec{p}$, and $\mathcal{Q}_2$ the quadrupole moment tensor. What is the leading order dependence of $V(\vec{x})$ on r for large r? (10)

Answer:

$$Q_T = \sum q_i = q + (-q) + q + (-q) = 0 \tag{17}$$

$$\vec{p} = \sum q_i \vec{x}_i = qd\hat{\imath} + (-q)d\hat{\jmath} + q(-d\hat{\imath}) + (-q)(-d\hat{\jmath}) = 0 \tag{18}$$

$$\begin{aligned} \mathcal{Q}_2 &= \sum q_i \frac{3\vec{x}_i\vec{x}_i - r_i^2\hat{I}}{2} \\ &= q\frac{3d^2\hat{\imath}\hat{\imath} - d^2\hat{I}}{2} - q\frac{3d^2\hat{\jmath}\hat{\jmath} - d^2\hat{I}}{2} + q\frac{3d^2\hat{\imath}\hat{\imath} - d^2\hat{I}}{2} - q\frac{3d^2\hat{\jmath}\hat{\jmath} - d^2\hat{I}}{2} \\ &= qd^2(\hat{\imath}\hat{\imath} - \hat{\jmath}\hat{\jmath}) \end{aligned} \tag{19}$$

Since Q_T and $\vec{p}$ vanish and $\mathcal{Q}_2$ does not, the leading order dependence in $V(\vec{x})$ for large r is quadrupole or $1/r^3$.

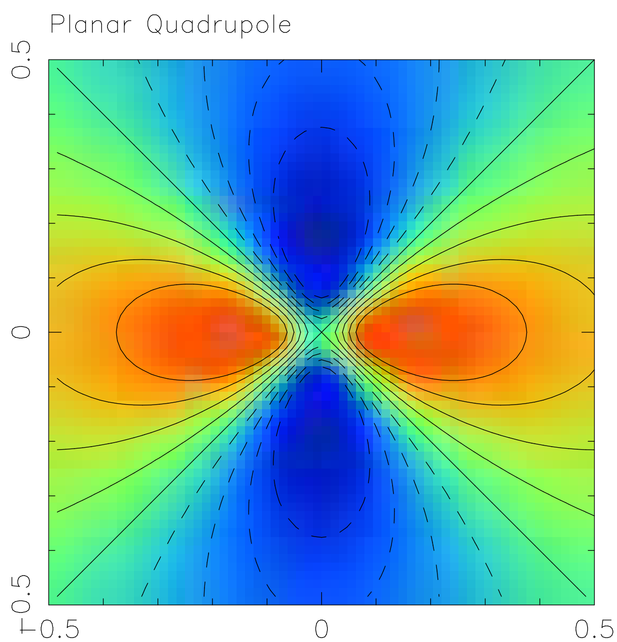
(b) Give the approximate expression (i.e. keep the lowest order multipole) for $V(r, \theta, \phi)$ knowing that $\hat{r} \cdot \hat{\imath} = \sin\theta\cos\phi$ and $\hat{r} \cdot \hat{\jmath} = \sin\theta\sin\phi$. (5)

Answer:

$$V(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r} \cdot \mathcal{Q}_2 \cdot \hat{r}}{r^3} = \frac{qd^2}{4\pi\epsilon_0}\frac{(\hat{r} \cdot \hat{\imath})(\hat{\imath} \cdot \hat{r}) - (\hat{r} \cdot \hat{\jmath})(\hat{\jmath} \cdot \hat{r})}{r^3} = \frac{qd^2 \sin^2\theta}{4\pi\epsilon_0}\left(\frac{\cos^2\phi - \sin^2\phi}{r^3}\right) \tag{20}$$

(c) Make a sketch of the equipotential surfaces in the $x - y$ plane indicating regions of positive and negative potential. (hint: it is not quite the same as the linear quadrupole but instead looks like a different hydrogen d orbital) (5)

Answer: I haven't made any special effort to adjust the vertical and horizontal scales. Colour is also indicating the potential, with blue low and red high.



Planar Quadrupole
0.5
0
-0.5
-0.5
0
0.5