Midterm Exam I: Electricity and Magnetism 322
October 22, 2003

Open text book with one formula sheet plus the magic triangle sheet. Point values are given with each question. Total exam is worth $100 + 10$ points. I will mark it out of 95. Maximum mark is still 100%.

1. Suppose that we have a scalar field that is given in Cartesian coordinates by

$$f(\vec{x}) = \frac{x}{a} \tag{1}$$

where a is some constant.

(a) Calculate ∇f. (5)

(b) Suppose that the Cartesian components of the gradient $(\nabla f)_i$ transform like a vector. Give the transformation law for $(\nabla f)'_i$ in Einstein summation notation if a_{ij} describes the rotation. The primes here refer to values in the transformed coordinate system. (5)

(c) Suppose that the rotation is by 45 degrees about the z-axis ($a_{11} = a_{22} = a_{12} = -a_{21} = \frac{\sqrt{2}}{2}$, $a_{33} = 1$, other elements are zero). Use the definition of a vector given in the previous question to find $(\nabla f)'_1$, $(\nabla f)'_2$, and $(\nabla f)'_3$. (10)

(d) In primed coordinates

$$f(\vec{x}) = \frac{\sqrt{2}}{2}\frac{(x' - y')}{a} \tag{2}$$

Calculate the components of the gradient with respect to these primed coordinates and show that the answer agrees with 1(c). (8)

2. The electric field in spherical polar coordinates is found to be

$$\vec{E}(\vec{x}) = \begin{cases} E_0 \frac{r}{a}\hat{r} & \text{for } r \le a \\ E_0 \frac{a^2}{r^2}\hat{r} & \text{for } r > a \end{cases} \tag{3}$$

where a is some constant.

(a) Sketch $\vec{E}(\vec{x})$ in the $x-y$ plane. The lengths of the vectors should reflect their magnitudes. Include the boundary $r = a$ using a dashed line. (5)

(b) Using Gauss's law in differential form find the charge density $\rho(\vec{x})$ in terms of E_0, r, a, and ϵ_0 in the two regions. In words, what does this charge distribution look like? (10)

(c) Determine $\oint \vec{E} \cdot d\vec{A}$ for a surface defined by a cube with edge length $a/2$ but offset from the origin by $a/10$ in the $\hat{\imath}$ direction. (5)

(d) Demonstrate that the scalar potential

$$V(\vec{x}) = \begin{cases} E_0 \frac{a^2 - r^2}{2a} + E_0 a & \text{for } r \le a \\ E_0 \frac{a^2}{r} & \text{for } r > a \end{cases} \tag{4}$$

gives the correct $\vec{E}$. (10)

(e) What is $\nabla \times \vec{E}$? (don't calculate, just state the answer and your reasons). (5)

(f) Show that V in the $r > a$ region satisfies Laplace's equation. (5)

3. BONUS: For the problem above give a value for total charge based on the form of V at large r presented in 2(d) (hint: notice that V takes the form of a point charge or a monopole at large r). (10)

4. A charge q sits on the z-axis distance d above the xy plane.

   (a) What is $\vec{x}\,'$? What is $r\,'$? (The primes now refer to the source coordinates.) (3)

   (b) You are interested in the potential at points along the x-axis distance x from the origin. What is $\vec{x} - \vec{x}\,'$? What is $|\vec{x} - \vec{x}\,'|$? (4)

   (c) What is the exact expression for $V(x, 0, 0)$ assuming $V(\infty) = 0$? Calculate V for $q = 10^{-9}$ C, $d = 0.01$ m, $x = 0.03$ m. Notice that $\frac{q}{4\pi\epsilon_0} = 9$ in MKS units. (5)

   (d) Now consider the multipole expansion of the potential. Don't be concerned that $N = 1$; this just makes the sums simpler! So $q_1 = q$ and $\vec{x}_1 = \vec{x}\,'$. Using equation 3.102 give the expressions for Q_T, $\vec{p}$, $\mathcal{Q}_2$ in terms of q, d, and the appropriate vectors and tensors. (6)

   (e) Using 3.91 and the expressions and the values above calculate the monopole, dipole, and quadrupole contributions to $V(\vec{x})$ at $\vec{x} = 0.03\,\hat{\imath}$ (metres). You should obtain 300, 0, and -17 V respectively. WARNING: r in the multipole expansion is not $|\vec{x} - \vec{x}\,'|$ it's just $|\vec{x}|$! (10)

   (f) Determine how good of an approximation this is by calculating the percentage difference between the sum of the first three terms of multipole expansion and the exact result. How do you expect this difference to change at larger r? (Even if you got stuck you should be able to answer this one using the numbers in the previous section.) (4)