Midterm Exam: Electricity and Magnetism 322
November 17, 2006

Closed book with one two-sided formula sheet and photocopy of front and back covers. Point values are given with each question. Be careful with your vectors and notation. Unless otherwise indicated state the assumptions you are making. Total midterm is worth 40 points.

1. (a) How do the electric field lines intersect the surface of a conductor? Why? (3)

   Answer: They intersect normal to the surface of a conductor. As to why, you might say that the tangential components of $\vec{E}$ must be continuous at a boundary (because $\nabla \times \vec{E} = 0$). Inside of the conductor $\vec{E} = 0$ and so $\vec{E}_{\text{tan,inside}} = 0$. Hence by continuity $\vec{E}_{\text{tan,outside}} = 0$. If the tangential components vanish, then the field must be normal to the surface. The other way to say this is that since $\vec{E} = -\nabla V$ we know that the equipotential surfaces are always normal to the field lines and the surface (and bulk) of a conductor is an equipotential.

   (b) A charge of q is at the location $(0,\ 0,\ d)$. If we use $V = 0$ at infinity as our reference give the formula for the potential in Cartesian coordinates. Does this potential satsify Laplace's equation or Poisson's equation? (3)

   Answer:
   $$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} \tag{1}$$
   This potential satisfies Poisson's equation since taking the Laplacian of it yields a delta-function at $z = d$, corresponding to the charge density ρ.

   (c) Now consider adding a grounded conductor at the $z = 0$ plane. Is the formula you gave in the previous question still a solution to given electrostatic problem? What part of the conditions we spoke about in the FISHTANK method does it violate? (3)

   Answer: No, it isn't a solution. It doesn't match the $V(x, y, z = 0) = 0$ boundary condition. It probably doesn't match the $V(x, y, z < 0)$ solution either. In the absence of any other charges we would just say that $V(x, y, z < 0) = 0$ is the solution for this region.

   (d) What is the technique used to solve the problem? Give the solution to the problem and its range of validity. Show mathematically how this has fixed our previous concern. (6)

   Answer: The technique used is *the method of images.* We imagine placing a charge of $-q$ at $z = -d$. This gives us the potential:
   $$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}} \tag{2}$$
   or in spherical polar coordinates
   $$V(r, \theta, \phi) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + d^2 + 2rd\cos\theta}}. \tag{3}$$
   These solutions are only valid for $z \geq 0$ or $\theta \leq \frac{\pi}{2}$ (the upper half plane).
   Just sub in $z = 0$ or $\cos\frac{\pi}{2} = 0$ and you can easily see that the terms cancel for all values of x, y, and r. So we have matched the $V = 0$ boundary condition at the plane.

2. In spherical polar coordinates the separable solutions to Laplace's equation are the products of Legendre polynomials $P_\ell(\cos\theta)$ and

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}} \tag{4}$$

   (a) Sketch the 3 lowest order Legendre polynomials as a function of u. (3)

   Answer: The functions are 1, u, and $\frac{1}{2}(3u^2 - 1)$ so they look like a constant, straight line through the origin, and a parabola. All of them intersect at (1,1). The domain of the functions is $[-1, 1]$. It is important to include the u scale.

   (b) Show that the function $V = ArP_{\ell=1}(\cos\theta)$ solves Laplace's equation in spherical polar coordinates. (A is a constant.) (3)

   Answer: $P_{\ell=1}(\cos\theta) = \cos\theta$ So the negative Laplacian operation on $V(r,\theta,\phi) = Ar\cos\theta$ looks like

$$
\begin{aligned}
& - \frac{1}{r^2}\cos\theta\frac{d}{dr}\left(r^2\frac{dr}{dr}\right) - \frac{r}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}(\cos\theta)\right) && (5)\\
&= -\frac{\cos\theta}{r^2}\frac{d}{dr}\left(r^2\right) - \frac{1}{r\sin\theta}\frac{d}{d\theta}\left(-\sin^2\theta\right) = -\frac{\cos\theta}{r^2}(2r) + \frac{1}{r\sin\theta}(2\sin\theta\cos\theta) && (6)\\
&= -\frac{2\cos\theta}{r} + \frac{2\cos\theta}{r} = 0 && (7)
\end{aligned}
$$

   (c) Given the form of V above, under what circumstances would it be most useful (i.e. what range of r, what symmetries in $u = \cos\theta$ and ϕ). (3)

Answer: The form above will diverge at large r so it would be most useful when $r < R$ where R is some limited size. It will also work right down to $r = 0$. The boundary conditions should be an odd function in u and show no ϕ dependence. As far as a concrete example this would be the leading term for the voltage dependence inside of two hemispheres with opposite voltage on each.

3. Consider the function

$$V(x,y,z) = A(x^2 + y^2 + 2z^2) \tag{8}$$

A is a constant.

   (a) Even without doing the calculation, why would you conclude this cannot be a solution to Laplace's equation? (3)

   Answer: This function has a local minimum at the origin. Solutions to Laplace's equation cannot have local minima or maxima.

   (b) What simple modification could you make to the sign(s) of the coefficients of x^2, y^2, and z^2 to give a solution to Laplace's equation. Demonstrate that it is a solution. (I think this is some constant multiplied by George!) (3)

   Answer: You can either change the sign of the z-coefficient or of both the x and y coefficients. If we choose to change the sign of the z-coefficient

$$
\begin{aligned}
-\nabla^2 V(x,y,z) &= -\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\left[A(x^2 + y^2 - 2z^2)\right] && (9)\\
&= -\frac{d^2}{dx^2}\left(Ax^2\right) - \frac{d^2}{dy^2}\left(Ay^2\right) + \frac{d^2}{dz^2}\left(2Az^2\right) && (10)\\
&= -2A - 2A + 4A = 0 && (11)
\end{aligned}
$$

(c) Even if you aren't so sure about the previous answer you can be sure it doesn't contain sines, cosines and exponentials of x, y, z. But we spent a lot of time talking about solutions to Laplace's equation that did. Why doesn't this one? (2)

Answer: The solutions to Laplace's equation in Cartesian coordinates that we discussed in class that contained sines, cosines, and exponentials were *separable* solutions so that $V(x,y,z) = X(x)Y(y)Z(z)$ was a product of the separate functions. This new solution is "separable" but not in the same way. It is a sum of the Cartesian functions (might be useful in some circumstances). It *is* separable in spherical polar coordinates. Rearrange the terms and use the substitutions $x^2 + y^2 + z^2 = r^2$ and $z^2 = r^2 \cos^2\theta$

$$V(x,y,z) = A(x^2 + y^2 + z^2) - 3Az^2 = Ar^2 - 3Ar^2\cos^2\theta \tag{12}$$

$$= -2Ar^2\left(\frac{3\cos^2\theta - 1}{2}\right) = -2Ar^\ell P_{\ell=2}(u = \cos\theta) \tag{13}$$

4. A parallel plate capacitor has plates parallel to the $x - y$ plane. The plates are separated by d and have total area A. The surface charge σ is independent of x and y. The plates carry equal and opposite charges. The potential function between the plates is $V = V_0 z/d$ and the electric field is $\vec{E} = -V_0/d\,\hat{z}$. Calculate the capacitance and the energy in terms of the given parameters by methods of your choosing. (You may even use the relationship between energy and capacitance. Remember the total volume enclosed by the capacitor is Ad.) (8)

Answer: There are several ways to do this problem and it should basically be a review of what you know about parallel plate capacitors. If you want to find capacitance directly from $C = Q/V$ you should know that in this definition it is assumed you have put charge Q and $-Q$ on the conductors and V is the potential difference between the conductors, not the scalar potential function itself. V between the plates is "given" as V_0 once you substitute $z = 0$ and $z = d$ into the potential *function*. We find σ_{upper} by using the boundary condition on $\vec{E}$. The normal vector points away from the conductor so is $-\hat{z}$ on the upper plate. $E_n(z = d) = \vec{E}(z = d)\cdot(-\hat{z}) = V_0/d$ and $\sigma_{\text{upper}} = \epsilon_0 V_0/d$. We find Q by multiplying by the area since σ_{upper} is constant. Note: suppose that you choose the lower plate instead and ended up with $-Q$. No problem, you still have Q in the definition; just drop the sign. If you don't like doing this just make sure you do the Q calculation on the conductor with positive Q (look at the direction of the field lines to determine which is which).

$$C = \frac{Q}{V} = \frac{\frac{\epsilon_0 A V_0}{d}}{V_0} = \frac{\epsilon_0 A}{d} \tag{14}$$

This is the standard result for a parallel plate capacitor which you should have memorized (or at least know right where to find it on your formula sheet).

To calculate U can use a few options:

$$U = \frac{1}{2}CV^2 = \frac{\epsilon_0 A V_0^2}{2d} \tag{15}$$

$$U = \frac{\epsilon_0}{2}\int \rho V\, d\tau = \frac{\epsilon_0}{2}\int_{\text{upper}} \sigma V_0\, da = \frac{\epsilon_0 A V_0^2}{2d} \tag{16}$$

$$U = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int \frac{V_0^2}{d^2} d\tau = \frac{\epsilon_0 A V_0^2}{2d} \tag{17}$$

Since the last method doesn't involve a calculation of σ you could use it to find U first and then use the first method to find C.