Solutions for Midterm Exam: Electricity and Magnetism 322
October 14, 2005

I didn't expect you to include all of the explanation that I have here.

1. (a) Give the definition of a vector in Einstein summation notation. Also give the dot product and cross product using Einstein summation notation. (4)

Answer: If A_i are the Cartesian components of a vector $\vec{A}$ and R_{ij} is the rotation matrix that describes the transformation between unprimed and primed coordinates then $\vec{A}$ is a vector if

$$A'_i = R_{ij} A_j. \tag{1}$$

We are using Einstein summation convention so repeated indices (j in this case) are summed from 1 to 3. The dot product (which gives a scalar) between vectors $\vec{A}$ and $\vec{B}$ with Einstein summation convention

$$\vec{A} \cdot \vec{B} = A_i B_i \tag{2}$$

and the i-th component of the cross product

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k \tag{3}$$

where ϵ_{ijk} is the Levi-Civita alternating tensor.

(b) If $\vec{x} = 3\hat{\imath} - 2\hat{\jmath} + \sqrt{3}\,\hat{k}$ what are x_1, x_2, and x_3? (2)

Answer: $x_1 = 3$, $x_2 = -2$, and $x_3 = \sqrt{3}$. Just a check to see if you understand the notation and prepare you for the next question.

(c) Consider a counter clockwise rotation of $\pi/6$ about the z-axis. The elements of the rotation matrix are $R_{11} = R_{22} = \sqrt{3}/2$, $R_{12} = -R_{21} = 1/2$, $R_{33} = 1$ and all other elements are equal to zero. What is x'_1? (4)

Answer: $\vec{x}$ is a vector (indeed, it is the prototype) so it obeys equation 1.

$$\begin{aligned} x'_1 &= R_{11}x_1 + R_{12}x_2 + R_{13}x_3 \\ &= \frac{\sqrt{3}}{2}(3) + \frac{1}{2}(-2) + (0)(\sqrt{3}) \\ &= \frac{3\sqrt{3} - 2}{2} \end{aligned} \tag{4}$$

(d) You perform a further rotation of 20 degrees counterclockwise about the new y-axis. What is $|\vec{x}\,''|$? (the two primes refer to the two transformations) (2)

Answer: This was a very difficult question if you failed to recognize that $|\vec{x}|$ is a scalar. It is of course a scalar and unaffected by rotations so

$$|\vec{x}\,''| = |\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{9 + 4 + 3} = 4. \tag{5}$$

2. (a) State Gauss' Law in integral and differential form. (2)

Answer:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \tag{6}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{7}$$

(b) According to Coulomb's Law the electric field of a point charge at the origin is

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{8}$$

Show that this is consistent with Gauss' Law in integral form if we choose a sphere of radius r centred on the origin as our Gaussian surface. Justify/very briefly explain any assumptions. (6)

Answer: On the sphere of radius r, $\vec{E}$ is a constant and according to equation 8 is $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$. The normal vector that gives the direction of the differential area element is $\hat{r}$ so that $d\vec{A} = \hat{r}dA$. Now we can evaluate the LHS of equation 6

$$\oint \vec{E}\cdot d\vec{A} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\oint_{sphere}\hat{r}\cdot\hat{r}dA = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\oint_{sphere}dA = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\epsilon_0} \tag{9}$$

since "q" is enclosed this is consistent with the RHS.

(c) Give the above expression for the electric field in Cartesian coordinates (can just state it). (2)

Answer:

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}} \tag{10}$$

(d) Give a sketch of the vector field in the $x-y$ plane (similar to what you have seen in class, not the tangent line kind) where the length of the vector at each location indicates the magnitude and direction of the vector field at that point. (2)

Answer: Looks like vectors emanating radially outward from the origin with a decrease in vector length as you move farther from the origin.

3. (a) What must $\nabla\times(-\nabla V)$ be equal to? Calculate it explicitly to demonstrate this fact for

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}} \tag{11}$$

using either Cartesian or spherical polar coordinates. (6)

Answer: $\nabla\times(-\nabla V)$ must be equal to 0 since it is the curl of a gradient. For this particular form

$$\begin{aligned} -\nabla V &= -\hat{r}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}(\ldots) - \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}(\ldots) \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \end{aligned} \tag{12}$$

(The 2nd and 3rd terms have no θ or ϕ dependence and they vanish.) Doing the same operation in Cartesian coordinates will give you equation 10.

Taking the curl in spherical polars, putting in zeroes for the θ and ϕ components.

$$\begin{aligned} \nabla\times\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\right) &= \frac{\hat{r}}{r\sin\theta}\left\{(\frac{\partial}{\partial\theta}(0) - \frac{\partial}{\partial\phi}(0)\right\} + \frac{\hat{\theta}}{r}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right) - \frac{\partial}{\partial r}(0)\right\} \\ &\quad + \frac{\hat{\phi}}{r}\left\{\frac{\partial}{\partial r}(r(0)) - \frac{\partial}{\partial\theta}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\right\} \\ &= 0 \end{aligned} \tag{13}$$

The non-zero terms above have neither θ nor ϕ dependence. In the calculation with Cartesian coordinates you do end up with non-zero terms but once you subtract ($\partial E_z/\partial y - \partial E_y/\partial z$, for example) the terms cancel each other.

(b) We know that $-\nabla^2 V = 0$ for $r \neq 0$. Using the Dirac delta function and your knowledge of the Green's function state what $-\nabla^2(\frac{1}{r})$ (or equivalently $-\nabla^2(\frac{1}{|\vec{x}|})$) is for all points, including the origin. (4)

Answer: From our knowledge of Green's function, $G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$, as pertains to Poisson's equation we know that

$$-\nabla^2 G = -\nabla^2 \left(\frac{1}{4\pi r}\right) = \delta^3(r) \tag{14}$$

when $\vec{x}' = \vec{0}$ Just multiplying by 4π we find that

$$-\nabla^2 \left(\frac{1}{r}\right) = 4\pi\delta^3(r). \tag{15}$$

4. Setup (with a diagram) but don't solve the Coulomb's Law formulation for the x-component of the electric field at $(x, 0, 0)$ of a charged line of uniform linear charge density λ that runs along the z-axis from 0 to ℓ. (Hint: the explicit expression for the x-component of $\vec{E}$ is $\hat{\imath} \cdot \vec{E}$) (8)

Answer: this is pretty much out of the book and notes (look there for the diagram). The only difference is that the integral runs from 0 to ℓ. The Coulomb's Law formulation says

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \tag{16}$$

and the integral is formally over all space as we integrate the source coordinates. The key first step is to *parametrize* the charge distribution. We can paramatrize a line with just one variable and taking the hint that it runs along the z-axis we will choose z' as our parameter and 0 to ℓ as our integration range (there is no charge anywhere else so that is why we don't integrate over all space). Other shapes and sizes of charge distributions would suggest different parametrization variables and different coordinate systems. The differential charge element $d^3x'\,\rho$ becomes $dz'\lambda$ for a uniformly charged line. We now need to choose a set of unit vectors to describe $\vec{x}$ and $\vec{x}'$ in terms of x, y, z, and z'. The problem states that the field point is on the x-axis so $\vec{x} = x\hat{\imath}$. The source points are all along the z-axis so $\vec{x}' = z'\hat{k}$. So the elements of the integrand are

$$\begin{aligned} \vec{x} - \vec{x}' &= x\hat{\imath} - z'\hat{k} && (17) \\ |\vec{x} - \vec{x}'| &= \sqrt{x^2 + z'^2}. && (18) \end{aligned}$$

If we just want E_x then $\vec{E} \cdot \hat{\imath}$ leaves $(\vec{x} - \vec{x}') \cdot \hat{\imath} = x$ in the numerator of the integrand. x and λ are not functions of z' so they come out of the integral.

$$E_x = \frac{\lambda}{4\pi\epsilon_0} x \int_0^\ell \frac{dz'}{(x^2 + z'^2)^{\frac{3}{2}}} \tag{19}$$