Advanced Macroeconomics, ECON 402
Introduction to Dynamic Optimization & Calculus of Variation

1 Why do we need to consider dynamics?

Thus far through your courses in Economics, you would have noticed that for all intents and purposes, classical calculus methods seem to suffice in aiding your solution search. The problems you have examined however tended to be for optimal choices within a fixed duration, which without loss of generality can be thought of as static optimization. If we are all short sighted, than perhaps considering dynamics is not necessary. However, given that economic agents such as yourself do consider the future, though to differing extents, an optimal solution in the short run does not imply they would be in the long run (Note that the fact we suffer from regret, does not mean that we are shortsighted, rather elements of the problem we face are stochastic in nature, and given any realization, we would feel we could have done things differently. But that remains after the fact. When our choices are viewed in terms of each problem in its entire sequences, it is likely optimal given our resources at the beginning of the problem.).

In *Dynamic Optimization*[1], we are finding answers to the question of optimal choices in each and every period of time within a certain planning horizon, which can be finite $[0, T]$ (in both continuous and discrete time) or infinite horizon $[0, \infty)$[2]. Such a solution would imply an *Optimal Time Path* in terms of every choice variable from the initial to the terminal period or the end of the planning horizon. Thus you can think of Dynamic Optimization as a multistage decision making process.

We will now briefly describe some of the key feature of Dynamic Optimization Problems:

1.1 Dynamic Optimization Problems

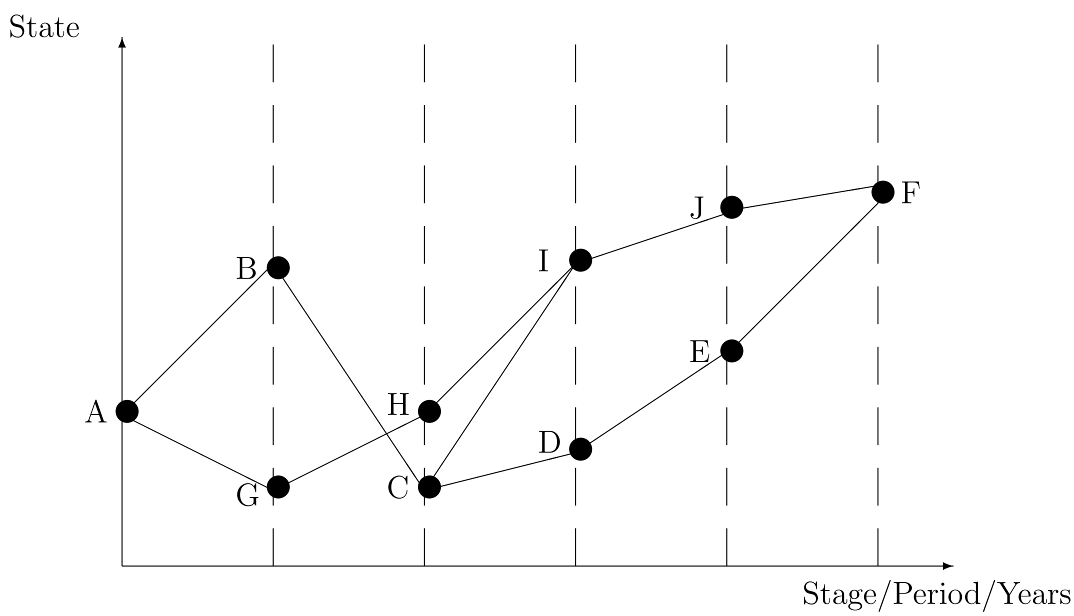
All dynamic problems consists of considerations of moving from one initial state of the world to a final state, for example in Growth Theory the initial state of an economy can

[1] For details for this set of notes, you should read Chiang (1992).

[2] We will discuss the nature of the problem in discrete time in greater detail subsequently in the course.

be thought of as when an economy begins life as a Republic post independence, and the objective of the social planner there is to take the economy towards the final state of development with Per Capita GDP comparable to those of OECD economies. In that sense, the Per Capita GDP is the measure of the *state* of the world. Such a process would necessarily have to take several periods. Such a process in discrete time is illustrated in figure 1.

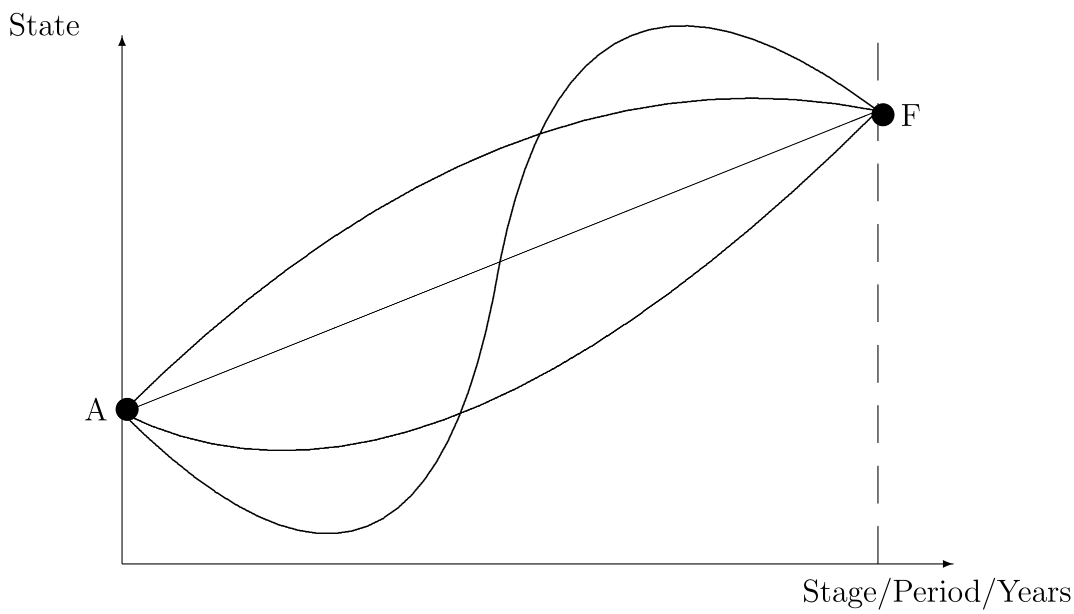
Figure 1: Multi-Stage Decision Process



Following the example, a developing economy might begin its transition towards developing status at the *initial state* A with a particular level of Per Capita GDP. It's ultimate objective being to arrive in *terminal state* F. The choices that would give rise to the path could include amount of expenditure on public infrastructure, education, etc against the level of institutional borrowing say from OECD nations or the IMF, or to encourage foreign direct investments. Each set of options would give rise to a sequence of paths, and an example of a path on figure 1 could be $A \rightarrow B$, and a sequence of arcs/lines could be that from $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F$ which could be generated from foreign direct investments. While an alternative process of development could be say generated by the government choosing to borrow mainly from wealthier nations to build it's social infrastructure. Such a policy may imply lower initial GDP, but may imply a smoother trajectory beyond the formative years. This trajectory could be depicted as $A \rightarrow G \rightarrow H \rightarrow I \rightarrow J \rightarrow F$. We call these points, A, B, ..., K *Vertices*. The sequence of arcs that yield the highest GDP would be the *Optimal Path*. Note that the paths could cross in the sense that for example at point C, the economy could correct it's trajectory by moving to I instead of D. In other words, the paths can cross. It is important to recognize that the path taken is that which *maximize or minimize the path values* (This path values could be in terms of the cost of achieving the terminal state, and which need not be monotonic.). Finally, in general a single static optimization procedure will not yield an optimal time path.

If instead, we are thinking about continuous time, the diagram of figure 1 is altered to the one below in figure 2. In continuous time, there are now an infinite number of *stages*, and an infinite number of *states*. Note that the straight line need not be the optimal plan because the choice depends on the cost of choosing that line, for example when we think about economic growth, there are cost involved in building infrastructure, providing education to the general populace and social cohesion which is hard to measure.

Figure 2: Decision Making under Continuous Time



For the rest of our discussion in the next few lectures, we will construe time as a continuous measure.

1.2 Functionals

Based on the above discussion, what we are trying to optimize is not so much an optimal choice per se, but optimal choices that gives rise to an optimal path. This means we are not trying to optimize a function, which by definition is a one-to-one onto mapping, but a mapping from paths to a real number (performance indices such as profits, costs, GDP per capita, etc).

There may be several paths that can be taken by an economy, and for the rest of our discussion, we will denote that path as $y_i(t)$ where $i \in \{1, 2, ...\}$ denotes the paths under consideration. This path values will change with time, and consequently are functions of time t. However, what are maximizing are functions of these paths which we denote as $V_i \equiv V(y_i(t))$, which are called *functionals*, in other words, these are the sequence of objective valuations dependent on the paths $y_i(t)$ as opposed to time t, which is an important distinction, and consequently they are known as *functionals* as opposed to *functions*. There is no reason to believe that the *functionals* under differing paths would yield the same values, in other words, the point F generated by each path need not be of the same value though they may reach the same point! Think about this!

In the prior discussion, we have assumed that our problem has a initial state/point and a terminal point. Framed in Economics, the former assumption is not strong, but the latter is. To that end, we should note now the numerous possibilities that we can assume for the terminal point.

1. Fixed Terminal Time T and Terminal State $y(T)$ such as in the initial example.

2. Fixed Terminal Time T but with Free Terminal State $y(T)$. This type of problem is commonly referred to as *Fixed Time Horizon Problem* or a *Vertical Terminal Line Problem* since time is fixed. Think of the problem faced by a incoming General Manager of a beleaguered sports franchise. Let's not name names here!

3. Variable Time T but with Fixed Terminal State $y(T)$. Such problems are commonly referred to as *Fixed Endpoint Problem* or a *Horizontal Terminal Line Problem*. In addition, should the objective of the planner be to achieve the Fixed Terminal State as quickly as possible without due concern to lifetime costs, in other words, a distinct preference for speed, the problem is alternatively known as *Time Optimal Problem*.

4. Finally, there is the case of when both Terminal Time and Terminal State are "Free", or more precisely described by an optimization constraint $F = \Phi(T)$ where F denotes the final state we had on figure 1 and 2. This is know as a *Terminal Curve Problem* or *Terminal Surface Problem*.

The above pertains to finite planning horizon problems. The problems associated with infinite planning horizon, $T \to \infty$ will be dealt with later in our discussion. For the moment, we will deal with the simple problem associated with finite horizon problems, and how there can be used to analyse macroeconomic issues.

1.3 Objective Functional

We now formally defined the objective functional in the continuous time case. To do so we need,

1. Initial Time/Stage,
2. Initial State, and
3. Direction of Path/Arc.

These three elements are t, $y(t)$, and $y'(t) = \frac{dy(t)}{dt}$. Let the value of the Path or Arc be denoted by the function $F \equiv F(t, y(t), y'(t))$, in other words, we assume such a function exists. Then the functional is just,

$$V(y) = \int_0^T F(t, y(t), y'(t))dt \tag{1}$$

As before note that the functional gets its value from the state variable $y \equiv y(t)$ and not time t itself. When we have two state variables y and z, the functional can be written as,

$$V(y, z) = \int_0^T F(t, y(t), z(t), y'(t), z'(t))dt \tag{2}$$

and it generalizes easily to n state variables. This is known as the *Standard Problem.*

However, there are other forms of functionals,

1. For problems that are not dependent on the actual path taken but solely on the location of the terminal state,

$$V(y) = G(T, y(T)) \tag{3}$$

Note that since there is no path, the functional is a function, and this is known as a *Terminal Control Problem*. Further, similar to the standard problem of equation (1) which may include more than one state variable, it is likewise true for this Terminal Control Problem, for example for two state variables y and z,

$$V(y,z) = G(T, y(T), z(T)) \tag{4}$$

2. When the problem is a composite of the two concerns of path taken, and terminal state (that is represented by equations (1) and (3)), we have,

$$V(y) = \int_0^T F(t, y(t), y'(t))dt + G(T, y(T)) \tag{5}$$

2 Calculus of Variations

The technique of Calculus of Variations was originally used to determine "the shape of a surface of revolution that would encounter the least resistance when moving through some resisting medium" (Chiang 1992). Problems such as the above can be represented as follows,

$$\begin{aligned} \text{max} \quad \text{or} \quad \text{min}\, V(y) &= \int_0^T F(t, y(t), y'(t))dt \\ \text{subject to} \qquad y(0) &= A \;\; (A \;\; \text{given}) \\ \& \qquad\qquad y(T) &= Z \;\; (Z \;\; \& \;\; T \;\; \text{given}) \end{aligned} \tag{6}$$

This is known as the *Fundamental Problem* of Calculus of Variations. Note that for a solution to the problem to exists, the functional must be integrable. The quest here is to find the optimal path from among all the admissible paths. Because it is related to calculus, the solution is dependent on our finding a first order condition, and you would likewise have to ensure you are maximizing or minimizing the objective functional through examining the second order conditions.

We will now derive the first order condition for the fundamental problem (6). The first order condition of the problem is known as the *Euler Equation*. To do so, first let $y^*(t)$ denote the optimal path that begins at time 0 and state A, and terminates at time T at state Z. We have to then examine how such a path differs from all other possible paths that fulfill those constraints at the initial and terminating points. Remember that since we are dealing with calculus, we will maintain the assumption that between the initial and terminal point, the path is continuous. Let $p(t)$ be a perturbation function at each period between time 0 and T that causes a deviation from the optimal path. To obtain all neighbouring paths, we multiple a small number ϵ to the perturbation function so that,

$$\begin{aligned} y(t) &= y^*(t) + \epsilon p(t) && (7) \\ \Rightarrow y'(t) &= y^{*\prime}(t) + \epsilon p'(t) && (8) \\ \Rightarrow \lim_{\epsilon \to 0} y'(t) &= y^{*\prime}(t) && (9) \end{aligned}$$

We know that the functional $V(y)$ is dependent on the path taken y. Each path we may consider now is generated by variation in ϵ, so that we can think of $V(y)$ as a *function* of ϵ, the perturbation factor, $V \equiv V(\epsilon)$. Since the functional achieves its optimal value at $y^*(t)$ when $\epsilon = 0$, the first order condition is defined by the derivative,

$$\left. \frac{dV(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0 \qquad (10)$$

which is a necessary condition for $y^*(t)$ to be indeed the optimal path. Note at this point that both ϵ and $p(t)$ are totally arbitrary.

To develop the first order condition:

1. First rewrite the functional of equation (1) in terms of (7) and (8),

$$\begin{aligned} V(\epsilon) &= \int_0^T F(t, y^*(t) + \epsilon p(t), y^{*\prime}(t) + \epsilon p'(t)) dt && (11) \\ \Rightarrow \frac{dV(\epsilon)}{d\epsilon} &= \int_0^T \left(\frac{\partial F}{\partial y(t)} \frac{\partial y(t)}{\partial \epsilon} + \frac{\partial F}{\partial y'(t)} \frac{\partial y'(t)}{\partial \epsilon} \right) dt && (12) \\ &= \int_0^T \left(F_y p(t) + F_{y'} p'(t) \right) dt = 0 \\ &= \int_0^T F_y p(t) dt + \int_0^T F_{y'} p'(t) dt = 0 && (13) \end{aligned}$$

where the derivatives of the integral makes use of the Leibniz's rule, and equation (13) is the necessary condition noted in equation (10). Nonetheless, this necessary condition is not operational since it still contains the arbitrary perturbation functions $p(t)$ and $p'(t)$.

2. Integrate the second integral of equation (13) by parts we obtain,

$$\begin{aligned}\int_0^T F_{y'}p'(t)dt &= F_{y'}p(t)|_0^T - \int_0^T p(t)\frac{dF_{y'}}{dt}dt \\ &= -\int_0^T p(t)\frac{dF_{y'}}{dt}dt\end{aligned} \tag{14}$$

where the last equality follows since at the initial and terminal time, all paths are the same in the standard problem of (6). Therefore the first order condition of equation (13) can be written as,

$$\int_0^T p(t)\left(F_y - \frac{dF_{y'}}{dt}\right)dt = 0 \tag{15}$$

3. To eliminate the final arbitrary term $p(t)$, note that for equation (15) to hold with equality, the arbitrary nature of $p(t)$ implies that need,

$$F_y - \frac{dF_{y'}}{dt} = 0 \;\; \forall t \in [0, T] \tag{16}$$

$$\Rightarrow F_y - (F_{y't} + F_{y'y}y'(t) + F_{y'y'}y''(t)) = 0 \;\; \forall t \in [0, T] \tag{17}$$

$$\Rightarrow F_{y'y'}y''(t) + F_{y'y}y'(t) + F_{y't} - F_y = 0 \;\; \forall t \in [0, T] \tag{18}$$

This is know as the *Euler Equation*, and that in general, it is a second order nonlinear differential equation. This means that it's solution will have two arbitrary constant terms (since you need to integrate to solve for the optimal path, and since this is a second order differential equation, the integral will create two constants). These constants can be found using the initial and terminal conditions in the fundamental problem of equation (6).

2.0.1 Examples

- Consider the following functional which is from Chiang (1992),

$$V(y) = \int_0^1 \left(ty + 2y'^2\right)dt$$

with $y(0) = 1$ and $y(1) = 2$. Using the Euler equation formula above and given the functional,

$$\begin{aligned} F_y &= t & F_{y'} &= 4y' \\ F_{y't} &= F_{y'y} = 0 & F_{y'y'} &= 4 \end{aligned}$$

So that the Euler equation is,

$$\begin{aligned} 4y''(t) - t &= 0 \\ \Rightarrow y''(t) &= \frac{t}{4} \\ \Rightarrow y'(t) &= \frac{t^2}{8} + \alpha \\ \Rightarrow y(t) &= \frac{t^3}{24} + t\alpha + \beta \end{aligned}$$

Using the initial and terminal points to define α and β, we get $\beta = 1$ and $\alpha = \frac{23}{24}$ so that the solution path is,

$$y(t) = \frac{t^3}{24} + t\frac{23}{24} + 1$$

- Consider the following functional which is from Chiang (1992),

$$V(y) = \int_0^2 \left(y^2 + t^2 y'\right) dt$$

with $y(0) = 0$ and $y(2) = 2$. Using the Euler equation formula above and given the functional,

$$\begin{aligned} F_y &= 2y & F_{y'} &= t^2 \\ F_{y'y'} &= F_{y'y} = 0 & F_{y't} &= 2t \end{aligned}$$

so that the Euler equation is,

$$\begin{aligned} 2t - 2y(t) &= 0 \\ \Rightarrow y(t) &= t \end{aligned}$$

2.1 Special Functional Cases

1. When the F function is free of y such that $F = F(t, y'(t))$, then $F_y = 0$, from equation (16) in the development of the Euler equation, we get $\frac{dF_{y'}}{dt} = 0$ which in turn implies that

$$F_{y'} = \text{constant} \tag{19}$$

2. When the F function is free of t such that $F = F(y(t), y'(t))$, from equation (18) the Euler equation is

$$F_{y'y'}y''(t) + F_{y'y}y'(t) - F_y = 0$$

Next multiply throughout the above by y',

$$\begin{aligned} y'(t)(F_{y'y'}y''(t) + F_{y'y}y'(t) - F_y) &= 0 \\ \Rightarrow \frac{d(y'F_{y'} - F)}{dt} &= 0 \end{aligned}$$

This in turn implies that,

$$F - y'F_{y'} = \text{constant} \tag{20}$$

which is a first order differential equation and in most instances is an easier Euler equation to work on.

3. When the F function depends solely on y' such that $F = F(y')$, from equation (18) the Euler equation becomes,

$$F_{y'y'}y''(t) = 0 \tag{21}$$

This equation is satisfied if either $y''(t) = 0$ or $F_{y'y'} = 0$. The first implies that $y(t)$ must be a straight line, and the same is true for the second.

4. When the F function is not dependent on y' such that $F = F(t, y)$, from equation (18) the Euler equation becomes $F_y = 0$ and is no longer a differential equation, and the problem is degenerate. The reason being since there are no arbitrary constants to define, there is no reason to believe the optimal path will begin and end at the initial and terminal points respectively except by sheer coincidence.

2.2 Transversality Conditions

As noted in section 1.2, when the initial and terminal states and time are given, the solution can be identified in the case of the typical solution to the standard problem, the second order differential equation. However, it is possible that the terminal state and time may themselves be variable, in which case we would need an additional condition to identify the solution. This condition is known as a *Transversality COndition.* Note that solution condition described here works in the case when the initial condition is variable as well.

2.3 General Transversality Condition

Given the variable terminal point, the Calculus of Variations problem is now,

$$\begin{aligned} \max \quad \text{or} \quad \min V(y) &= \int_0^T F(t, y(t), y'(t))dt \qquad (22) \\ \text{subject to} \qquad y(0) &= A \quad (A \ \text{ given}) \\ \& \qquad y(T) &= y_T \quad (y_T \ \ \& \ \ T \ \text{ variable}) \end{aligned}$$

In this problem, the issue is not only to choose the optimal path, but to choose the optimal terminal state and time as well. As in the development of the Euler Equation, let $p(t)$ be the perturbation function, and ϵ be the perturbation factor, both being arbitrary.

Let T^* be the optimal terminal period, then we can denote all other possible terminal period as,

$$T = T^* + \epsilon \Delta T \qquad (23)$$

where ΔT is an arbitrary fixed change in terminal time T. Since ΔT is fixed then $T \equiv T(\epsilon)$, which in turn implies that $\frac{dT}{d\epsilon} = \Delta T$. In addition, as before to obtain other trajectories,

$$\begin{aligned} y(t) &= y^*(t) + \epsilon p(t) \qquad (24) \\ \Rightarrow y'(t) &= y^{*\prime}(t) + \epsilon p'(t) \qquad (25) \end{aligned}$$

However, although $p(0) = 0$ at the initial time, in other words the initial point remains fixed, $p(T) = 0$ can no longer hold since terminal state and time are variable. The functional can be written as,

$$V(\epsilon) = \int_0^{T(\epsilon)} F(t, y^*(t) + \epsilon p(t), y^{*\prime}(t) + \epsilon p'(t)) \qquad (26)$$

Thus to derive the general transversality condition,

1. First differentiating the functional $V(\epsilon)$ with respect to ϵ at $\epsilon = 0$ as in our development of the Euler Equation,

$$\left.\frac{dV}{d\epsilon}\right|_{\epsilon} = \int_0^{T(\epsilon)} \left.\frac{dF}{d\epsilon}\right|_{\epsilon=0} + F(T, y(T), y'(T)) \left.\frac{dT}{d\epsilon}\right|_{\epsilon=0} = 0 \qquad (27)$$

where the derivative is obtained using the Leibniz's rule. As in the development of the Euler Equation, the following obtains

$$\begin{aligned}\int_0^T F_y p(t)dt + \int_0^T F_{y'} p'(t)dt + [F]_{t=T}\Delta T &= 0 \\ [F_{y'}]_{t=T}\, p(T) + \int_0^T p(t)\left(F_y - \frac{dF_{y'}}{dt}\right)dt + [F]_{t=T}\Delta T &= 0 \qquad (28)\end{aligned}$$

Notice that we have the first term unlike in our development of the Euler equation because although $p(0) = 0$, $p(T) \neq 0$. In addition, see that each of the three terms have its own arbitrary term $p(T)$, $p(t)$ and ΔT respectively, so that we cannot use the technique in the Euler Equation development as before, but rather each has to be set equal to zero independently.

As earlier, first notice that when the second term is set equal to zero, we obtain the Euler Equation of before which implies that *the Euler Equation remains a necessary condition in the variable endpoint problem.* It is thus in the first and third term that the transversality condition lie.

2. To obtain the transverality condition, we have to connect the two arbitrary terms $p(T)$ and ΔT, to that end notice that the element which characterizes the change in optimal path is changes in the terminal state y_T, Δy_T. Then all we need to do is to characterize Δy_T in terms of the two arbitrary terms. Note first that for any perturbation of the optimal path at time T, y_T changes by the factor $p(T)$ at T. However, this presumes there is no change in time. In the general case on hand, both the state and time are variable. The rate of change of y_T is nonetheless just the rate of change of the path at T, $y'(T)$ multiplied by the amount of change in T, Δ. In other words,

$$\begin{aligned}\Delta y_T &= p(T) + y'(T)\Delta T \\ \Rightarrow p(T) &= \Delta y_T - y'(T)\Delta T \qquad (29)\end{aligned}$$

3. Now substituting equation (29) into equation (28) and dropping the second term (associated with the Euler equation), since the latter we know it must be zero, to get,

$$\begin{aligned}[F_{y'}]_{t=T}\,(\Delta y_T - y'(T)\Delta T) + [F]_{t=T}\Delta T &= 0 \\ [F - y'(T)F_{y'}]_{t=T}\,\Delta T + \Delta y_T\,[F_{y'}]_{t=T} &= 0 \qquad (30)\end{aligned}$$

which is the general transversality condition associated with a particular time T which consequently replaces the terminal time condition when the terminal point is variable. Depending on the case being considered as noted earlier, the general transversality condition can be altered for the case, specifically,

(a) **Vertical Terminal Line/Fixed Time Horizon Problem**: For this case, $\Delta T = 0$ so that $\Delta y_T \left[F\right]_{t=T}$ remains. However, Δy_T is arbitrary so that the transversality condition here becomes,

$$\left[F_{y'}\right]_{t=T} = 0 \tag{31}$$

which is also known as the *Natural Boundary Condition.* If we think of F as a social welfare function (a firm's profit function can be construed as likewise a firm(social) welfare function), the above condition says that at the terminal time, the optimal choice path chosen for the state variable should not yield any additional gains to be reaped beyond the terminal time T.

(b) **Horizontal Terminal Line/Fixed Endpoint Problem**: For this case, $\Delta y_T = 0$ so that $\left[F - y'(T)F_{y'}\right]_{t=T} \Delta T$ remains in the transversality condition. As above, ΔT is arbitrary so that for the condition to hold we would need,

$$\left[F - y'(T)F_{y'}\right]_{t=T} = 0 \tag{32}$$

Using the same social welfare function interpretation of the prior condition, the interchange of the current and future choices in the distribution of the state variable choices across time must be completed when the terminal point is reached.

(c) **Terminal Curve**: We know for the terminal curve problem $y_T = \Phi(T)$, which helps in identifying the solutions to y_T and T, and the other equation is provided by the transversality condition. To make the transversality condition operational, note that the relationship between y_T and T implies that $\Delta y_T = \Phi'(T)\Delta T$ so that the general transversality condition becomes,

$$\begin{aligned}
\left[F - y'(T)F_{y'}\right]_{t=T} \Delta T + \Phi'(T)\Delta T \left[F_{y'}\right]_{t=T} &= 0 \\
\Rightarrow \left[F - y'(T)F_{y'} + \Phi' F_{y'}\right]_{t=T} \Delta T &= 0 \\
\Rightarrow \left[F + \left(\Phi' - y'\right) F_{y'}\right]_{t=T} \Delta T &= 0 \\
\Rightarrow \left[F + \left(\Phi' - y'\right) F_{y'}\right]_{t=T} &= 0 \qquad (33)
\end{aligned}$$

where the last equality holds since ΔT is arbitrary, and thus the two unknowns at the terminal point can be defined.

3 Optimal Combination of Unemployment and Inflation

The following application is drawn from Chiang (1992), which is a modification of Taylor (1989). You know from your understanding of the Phillips curve that there is a trade off between having low inflation rates and unemployment faced by an economy. Let the ideal income level of an economy be Y^* when inflation rate is 0. Any deviation from this ideal is considered a loss to society. Denote the realized income as Y, and the inflation rate as p. Define the social loss function as,

$$L = (Y - Y^*)^2 + \alpha p^2 \tag{34}$$

where $\alpha > 0$. Note that because the deviations from the desired levels of unemployment/income and inflation are squared, the greater the deviation from the desired levels, Y^* and 0 respectively, the greater is the penalty. Relative to each other, deviations of unemployment/income and inflation, the weights are in the ration of $\frac{1}{\alpha}$.

Let π denote the expected rate of inflation, so that the expectations-augmented Phillips tradeoff between $(Y^* - Y)$ and p is captured by,

$$p = \beta(Y - Y^*) + \pi \tag{35}$$

where β is a positive constant. Equation (35) thus say that if economic activity is greater than that desired, actual inflation p will be greater than that expected, π.

Let the formation of inflation expectation be adaptive so that,

$$\frac{d\pi}{dt} = \pi' = \psi(p - \pi) \tag{36}$$

where $\psi \in (0, 1]$. Equation (36) says that if actual inflation is greater than expected, then expected future inflation will rise, and vice versa, consequently the adaptive nature of inflation expectation.

Combining equations (35) and (36), we get,

$$
\begin{aligned}
\pi' &= \beta\psi(Y - Y^*) \\
\Rightarrow (Y - Y^*) &= \frac{\pi'}{\beta\psi} && (37) \\
\Rightarrow p &= \frac{\pi'}{\psi} + \pi && (38) \\
\Rightarrow L(\pi, \pi') &= \left(\frac{\pi'}{\beta\psi}\right)^2 + \alpha\left(\frac{\pi'}{\psi} + \pi\right)^2 && (39)
\end{aligned}
$$

where the last equation is the Social Loss Function.

3.1 Fundamental Problem Case

Let the initial expected inflation be π_0, and the terminal value be 0. In addition, let the discount factor be ρ. The social planner's problem then is to determine the optimal path of expected inflation, π,

$$
\begin{aligned}
\min \mathcal{L}(\pi, \pi') &= \min \int_0^T L(\pi, \pi')e^{-\rho t} dt \\
&= \min \int_0^T \left\{\left(\frac{\pi'}{\beta\psi}\right)^2 + \alpha\left(\frac{\pi'}{\psi} + \pi\right)^2\right\} e^{-\rho t} dt && (40)
\end{aligned}
$$

subject to,

$$
\begin{aligned}
\pi(0) &= \pi_0 > 0 \\
\pi(T) &= 0
\end{aligned}
$$

Given the functional's form, we know

$$
\begin{aligned}
F_\pi &= 2\alpha\left(\frac{\pi'}{\psi} + \pi\right) e^{-\rho t} \\
F_{\pi'} &= 2\left\{\frac{\pi'}{\beta^2\psi^2} + \frac{\alpha}{\psi}\left(\frac{\pi'}{\psi} + \pi\right)\right\} e^{-\rho t} \\
&= 2\left\{\frac{1 + \alpha\beta^2}{\beta^2\psi^2}\pi' + \frac{\alpha\pi}{\psi}\right\} e^{-\rho t} \\
F_{\pi' t} &= -2\rho\left\{\frac{1 + \alpha\beta^2}{\beta^2\psi^2}\pi' + \frac{\alpha\pi}{\psi}\right\} e^{-\rho t} \\
F_{\pi'\pi} &= \frac{2\alpha e^{-\rho t}}{\psi} \\
F_{\pi'\pi'} &= \frac{2(1 + \alpha\beta^2)}{\beta^2\psi^2} e^{-\rho t}
\end{aligned}
$$

Consequently using the Euler Equation formula of equation (18),

$$\begin{aligned}
F_{\pi'\pi'}\pi'' + F_{\pi'\pi}\pi' + F_{\pi't} - F_{\pi} &= 0 \\
\Rightarrow \frac{(1+\alpha\beta^2)}{\beta^2\psi^2}\pi'' + \frac{\alpha}{\psi}\pi' - \rho\left(\frac{1+\alpha\beta^2}{\beta^2\psi^2}\pi' + \frac{\alpha\pi}{\psi}\right) - \alpha\left(\frac{\pi'}{\psi} + \pi\right) &= 0 \\
\Rightarrow \frac{(1+\alpha\beta^2)}{\beta^2\psi^2}\pi'' - \frac{(1+\alpha\beta^2)\rho}{\beta^2\psi^2}\pi' - \frac{\alpha(\rho+\psi)}{\psi}\pi &= 0 \\
\Rightarrow \pi'' - \rho\pi' - \frac{\beta^2\psi\alpha(\rho+\psi)}{(1+\alpha\beta^2)}\pi &= 0 \qquad (41)
\end{aligned}$$

It is clear then that equation (41) is a homogeneous second order differential equation, and the solution consists of only the *Complementary Function*[3]. Let $\frac{\beta^2\psi\alpha(\rho+\psi)}{(1+\alpha\beta^2)}$ be τ. The solution to the differential equation is thus of the form,

$$\pi^*(t) = \theta_1 e^{r_1 t} + \theta_2 e^{r_2 t} \qquad (42)$$

where r_1 and r_2 are the characteristic roots $\frac{1}{2}\left(\rho \pm \sqrt{\rho^2 + 4\tau}\right)$, one positive and the negative negative. Let $r_1 > 0$ and $r_2 < 0$. To find θ_1 and θ_2, use the initial and terminal state values,

$$\begin{aligned}
\theta_1 + \theta_2 &= \pi_0 \\
\theta_1 e^{r_1 T} + \theta_2 e^{r_2 T} &= 0
\end{aligned}$$

Solving the simultaneous equations,

$$\begin{aligned}
(\pi_0 - \theta_2)e^{r_1 T} + \theta_2 e^{r_2 T} &= 0 \\
\Rightarrow \theta_2 &= \frac{\pi_0 e^{r_1 T}}{e^{r_1 T} - e^{r_2 T}} > 0 \qquad (43) \\
\Rightarrow \theta_1 &= \pi_0 - \frac{\pi_0 e^{r_1 T}}{e^{r_1 T} - e^{r_2 T}} \\
&= \frac{-\pi_0 e^{r_2 T}}{e^{r_1 T} - e^{r_2 T}} < 0 \qquad (44)
\end{aligned}$$

Since both components to the solutions are decreasing function, this implies that the trajectory of the optimal path is decreasing and is bounded on the top by $\theta_2 e^{r_2 T}$ and the bottom by $\theta_1 e^{r_1 T}$. This can be verified by differentiating the optimal path by t,

$$\pi^{*\prime}(t) = r_1\theta_1 e^{r_1 t} + r_2\theta_2 e^{r_2 t} < 0$$

[3]For more details on differential equations see Chiang (1984) and Chiang (1992)

since θ_1 and r_2 are both less than 0. This trajectory for expected inflation can be achieved through affecting income and actual inflation in the following manner; based on equation (37) $Y - Y^* < 0$, in other words income realization must fall below the desired level Y^*, and based on equation (38), the actual price level will fall if the rate of decline in the expected inflation rate π' if less than the expected inflation rate $\psi\pi$.

3.2 Variable Terminal Point Case

It is interesting to frame the above question on unemployment and inflation to one with a terminal line as opposed to a terminal point $\pi(T) = 0$. Much of the solution discussed remains with the exception of the definition of the constants θ_1 and θ_2, so that as before,

$$\theta_1 + \theta_2 = \pi_0$$

From the transversality condition of equation (31)

$$F_{\pi'}\big|_{t=T} = 2\left\{\frac{1+\alpha\beta^2}{\beta^2\psi^2}\pi' + \frac{\alpha\pi}{\psi}\right\}e^{-\rho t}\bigg|_{t=T} = 0 \tag{45}$$

which implies that we would need,

$$\begin{aligned} \frac{1+\alpha\beta^2}{\beta^2\psi^2}\pi' + \frac{\alpha\pi}{\psi}\bigg|_{t=T} &= 0 \\ \Rightarrow \pi'(T) + \frac{\alpha\beta^2\psi}{1+\alpha\beta^2}\pi(T) &= 0 \end{aligned} \tag{46}$$

Let the coefficient of $\pi(T)$ be ϕ. We already know that,

$$\begin{aligned} \pi^*(t) &= \theta_1 e^{r_1 t} + \theta_2 e^{r_2 t} \\ \Rightarrow \pi^{*\prime}(t) &= r_1\theta_1 e^{r_1 t} + r_2\theta_2 e^{r_2 t} \end{aligned}$$

so that the transversality condition can be written as,

$$\begin{aligned} r_1\theta_1 e^{r_1 T} + r_2\theta_2 e^{r_2 T} + \phi(\theta_1 e^{r_1 T} + \theta_2 e^{r_2 T}) &= 0 \\ \Rightarrow (r_1 + \phi)\theta_1 e^{r_1 T} + (r_2 + \phi)\theta_2 e^{r_2 T} &= 0 \end{aligned} \tag{47}$$

We can now use equation (47) and the initial point condition to solve for the constants θ_1 and θ_2 which are,

$$
\begin{aligned}
0 &= (r_1 + \phi)(\pi_0 - \theta_2)e^{r_1 T} + (r_2 + \phi)\theta_2 e^{r_2 T} \\
\Rightarrow \theta_2 &= \frac{\pi_0 (r_1 + \phi)e^{r_1 T}}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \qquad (48) \\
\Rightarrow \theta_1 &= \pi_0 - \frac{\pi_0 (r_1 + \phi)e^{r_1 T}}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \\
&= \frac{-\pi_0 (r_2 + \phi)e^{r_2 T}}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \qquad (49)
\end{aligned}
$$

which gives the optimal terminal state at terminal time T to be,

$$
\begin{aligned}
\pi^*(T) &= \frac{\pi_0 e^{(r_1 + r_2)T}}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \{-(r_2 + \phi) + (r_1 + \phi)\} \\
&= \frac{\pi_0 e^{(r_1 + r_2)T}(r_1 - r_2)}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \\
&= \frac{-\pi_0 e^{2\rho T}\left(\sqrt{\rho^2 + 4\tau}\right)}{(r_1 + \phi)e^{r_1 T} - (r_2 + \phi)e^{r_2 T}} \neq 0 \qquad (50)
\end{aligned}
$$

that is when the terminal expected inflation rate is free, the expected inflation rate at the terminal period T is in fact negative, which is interesting compared to the constrained expected inflation of 0 in the earlier case. What do you think this implies for the actual inflation rate, and economic activity? You should note that a principal problem with such an approach to macroeconomic modelling is that the level of economic activity and inflation rate are not tied to their relationship through the operation within the economy.

4 Infinite Horizon & Constrained Problems

4.1 Infinite Horizon Problems

For most scenarios associated with the typical individual agent, even the "Politician", the finite planning horizon should amply mirror their true decision making process. However, from the perspective of a true social planner contemplating the welfare of the current and future generations, for example someone advising the government, the finite horizon framework places restrictions on their concerns, consequently an infinite horizon model would be more apt. In using a infinite horizon model, we create problems both in methodology, as well as the fact that our parameters of the optimal path may no longer be constant throughout. The two main methodological issue are that the functional may not be convergent (In the sense that the integral is from $t \in [0, \infty)$ as opposed to $t \in [0, T]$ consequently the integral is an improper integral and may diverge. If it diverges, there may then exist more than one path which gives rise to an infinite value to the objective functional.), in other words integral may not have a finite value, as well as the issue of the transversality condition.

The following are some conditions that sufficient for convergence of the functional,

- **Condition 1**: Given $\int_0^\infty F(t, y, y')dt$, if F is finite on some interval, and if it takes on the value of 0 from $t \in [\bar{t}, \infty)$, the integral will converge. In other words, although the integral is an improper integral, the implicit upper bound $\bar{t}$ changes it to a proper integral.

  Note that the condition that $F \to 0$ as $t \to \infty$ is neither an necessary or sufficient condition for convergence.

- **Condition 2**: If $F(t, y, y') \equiv G(t, y, y')e^{\rho t}$ where ρ is some positive rate of discount common in economic analysis, and if the G function is bounded from the above, the integral will converge. The intuition is as follows, the discount factor and the exponential function causes the entire function to tend towards 0, either if the effect that t has on G is slower, or particularly when G has an upper bound say $\overline{G}$, thereby forcing the entire function F to converge as $t \to \infty$. In other words,

$$\int_0^\infty G(t, y, y')e^{-\rho t}dt \leq \int_0^\infty \overline{G}e^{-\rho t}dt = \frac{\overline{G}}{\rho} \tag{51}$$

4.2 Transversality Condition

There has been some controversy over the use of Calculus of Variations problems to infinite state problems, particularly pertaining to the use of transversality conditions which has been shown not to be applicable in the infinite horizon problem. Nonetheless, we will provide a short brief on the transversality conditions. The conditions are not unlike the general transversality conditions we derived for the finite horizon problem, specifically,

$$\left[F - y'(T)F_{y'}\right]_{t \to \infty} \Delta T + \Delta y_T \left[F_{y'}\right]_{\to \infty} = 0 \tag{52}$$

Since both ΔT and Δy_T are arbitrary, each of the individual coefficients to the two terms must tend to 0. That is the transversality condition for the fact that the problem has infinite horizon implies that,

$$\lim_{t\to\infty} [F - y'(T)F_{y'}] = 0 \tag{53}$$

and the transversality condition if there is no terminal state given for the problem,

$$\lim_{t\to\infty} [F_{y'}] = 0 \tag{54}$$

If on the other hand the asymptotic terminal state is specified or found to be some constant y_∞, Δy_T will eventually disappear and the last transversality condition would not be necessary. This means formally that,

$$\lim_{t\to\infty} y(t) = y_\infty \tag{55}$$

Note that when the last equation is true, there is no controversy in the application of Calculus of Variations to an infinite horizon problem.

4.3 Constrained Problems

The discussion thus far has not explicitly handled constraints and their effects or alteration to the optimization problem, and how it can be handled. Using the last economic example regarding the trade off between unemployment and inflation rate, you would have noticed that the manner in which expectation is updated was said to be adaptive. That in effect is a constraint on the social loss function being optimized. The manner it was handled was through substitution into the functional. However, we can likewise solve the problem using the Lagrange Multipliers method. The application will to this problem will be treated as an exercise for you to verify. We will now discuss the various forms of constrains and how they can be included.

- **Equality Constraints**: Suppose the problem is,

$$V = \int_0^T F(t, y_1, ..., y_n, y_1', ..., y_n') \tag{56}$$

$$\text{subject to} \quad \begin{aligned} g^1(t, y_1, ..., y_n) &= c_1 \\ &\vdots \\ g^m(t, y_1, ..., y_n) &= c_m \end{aligned} \tag{57}$$

where g^i $i \in \{1, ..., m\}$ are independent constraint functions, and c_i are constants. The independence of the constraint functions implies that the $m \times m$ Jacobian exists and is none zero. Further, the number of constraints $m < n$ where n is the number of state variables. The reason being if $m = n$, the set of constraint equations would then uniquely determine the n number of $y_i(t)$ paths, and their would be no optimization problem. This means that at the least, there should be at least 2 state variables, to accommodate a single constraint.

To solve this constrained problem, we use the Lagrange Multiplier method to convert the constrained problem into an unconstrained problem. First convert the integrand,

$$\mathcal{F} = F + \sum_{i=1}^{m} \lambda_i(t)(c_i - g^i) \tag{58}$$

Note the key difference between the typical use of the Lagrange method in the static optimization case, and the current case. Firstly, the constraints and the shadow prices ($\lambda_i(t)$) are added to the integrand and not to the objective functional. Secondly, the shadow prices or multipliers are not constants but a function of time since the constraints need to be satisfied for the entire duration of the planning horizon.

The new functional now is,

$$\mathcal{V} = \int_0^T \mathcal{F} dt \tag{59}$$

Thus we have converted the constrained problem into an unconstrained one. Note that when all the constraints are satisfied, $V = \mathcal{V}$. Why? When optimizing the new functional, as in the static case, optimize with respect to all the state variables and t, but in addition, treat the multipliers as state variables as well. We know that optimizing the functional we get the Euler-Lagrange Equation similar to the Euler Equation we obtained in equation (16) (Think of $\mathcal{F}$ as if it were F, and λ_i as another set of state variables noted prior), so that for the usual state variables,

$$\mathcal{F}_{y_j} - \frac{d\mathcal{F}_{y'_j}}{dt} = 0 \tag{60}$$

$\forall j \in \{1, 2, ..., n\}$and similarly for the multipliers,

$$\begin{aligned}
\mathcal{F}_{\lambda_i} - \frac{d\mathcal{F}_{\lambda'_i}}{dt} &= 0 \\
\Rightarrow \mathcal{F}_{\lambda_i} &= 0 \quad (61) \\
\Rightarrow c_i - g^i &= 0 \quad (62)
\end{aligned}$$

the second equality is because $\mathcal{F}$ is not a function of λ'_i, and the third equality shows why $V = \mathcal{V}$ when the Lagrange Functional is optimized with all constraints satisfied. Note that to completely solve this constrained finite horizon problem we would in addition require the initial and terminal conditions for all the state variables or to obtain the transversality conditions.

- **Differential Equations Constraints**: When the constraints are themselves differential equations, the problem becomes,

$$V = \int_0^T F(t, y_1, ..., y_n, y'_1, ..., y'_n) \quad (63)$$

$$\text{subject to} \quad \begin{aligned}
g^1(t, y_1, ..., y_n, y'_1, ..., y'_n) &= c_1 \\
&\vdots \\
g^m(t, y_1, ..., y_n, y'_1, ..., y'_n) &= c_m
\end{aligned} \quad (64)$$

with the appropriate boundary constraints included in the problem as well. This problem adds nothing new to what you have learned above since the Euler-Lagrange equations with respect to the state variables, and the multipliers remain the same.

- **Inequality Constraints**: When the problem involves inequality constraints, we have

$$\max V = \max \int_0^T F(t, y_1, ..., y_n, y'_1, ..., y'_n) \quad (65)$$

$$\text{subject to} \quad \begin{aligned}
g^1(t, y_1, ..., y_n, y'_1, ..., y'_n) &\leq c_1 \\
&\vdots \\
g^m(t, y_1, ..., y_n, y'_1, ..., y'_n) &\leq c_m
\end{aligned} \quad (66)$$

including the boundary constraints as well. As before, the solution to sorts of problem remains the same. Note further that since the constraints are inequalities, that is the constraints need not be binding, there is no real concern in the number of constraints exceed the number of state variables under consideration. However, to ensure that $V = \mathcal{V}$, we need a complementary slackness condition to be coupled with the Euler-Lagrange equation for the multipliers,

$$\lambda_i(t)(c_i - g^i) = 0 \tag{67}$$

so that if the constraint is binding, we are back to the equality problem, and if the constraint is not binding (in other words the constraint hold with a strict inequality) that the multiplier must hold at the value of 0 for all periods under consideration.

- **Integral Constraints/Isoperimetric Problem**: The solution method can also handle problem when the constraints themselves are integrals, but for the sake of brevity, we will not discuss this in this class. However, should you be curious, you can find out how such problems can be solved from Chiang (1992).

5 Frank Ramsey's Theory of Saving

Ramsey (1928) examined the optimal social savings behavior of an economy using the Calculus of Variations, and it remains one of the most important papers on optimal economic growth. The central problem he was examining was the optimal level of resource allocation that would permit optimal economic growth, specifically how much should an economy should consume in any current period, and how much should be invested so as to increase future production capacity.

The model makes the following assumptions:

1. Output Q is assumed to be a function of capital K and labour L, in other words $Q \equiv Q(K, L)$, and this technology is assumed to be time invariant, so that he's model assumes that there is no technological progress, an assumption we will relax later in this course.

2. Capital does not depreciate.

3. Population size is stationary, in other words neither increasing or decreasing.

4. Labour is a state variable, and its use incurs a societal disutility $D(L)$ with increasing marginal disutility, $D_{LL} \geq 0$

5. Output not consumed will be saved (S) and result in investment and consequently capital accumulation K'. That is $Q = C + S = C + K'$ or more precisely,

$$C = Q(K, L) - K' \tag{68}$$

Thus capital K is a state variable.

6. The production function is an increasing function in all its variables, and that diminishing marginal product of capital and labour hold, in other words, $Q_{KK} \leq 0$ and $Q_{LL} \leq 0$.

7. The social welfare function is dependent on consumption alone, $U \equiv U(C)$ with decreasing marginal utility, $U_{CC} \leq 0$.

8. The net social welfare function is thus $U(C) - D(L)$. Consequently, C and L are functions of time, and this is true for Q and K indirectly as well.

The social planner's problem is thus to maximize the net social welfare function for all generations of constituents,

$$\max \int_0^{\infty} \{U(C) - D(L)\}dt \tag{69}$$

$$\Rightarrow \max \int_0^{\infty} \{U(C = Q(K, L) - K') - D(L)\}dt \tag{70}$$

so that as noted before, the problem has two state variables, K and L. However, notice that the functional is not dependent on L' so that the problem for the optimal level of labour input is degenerate since all we need to do is to find the maxima and keep it at that level for the entire planning horizon.

You should notice that the functional does not have a discount factor, which implies that in and of itself, there is no specific reason the integrand or the functional would converge. The principal reason that Ramsey (1928) eliminated the discount factor was because he did not see fit that a social planner could or should discount the welfare of future generations in favour of the current. Consequently, to bring some closure to the convergence problem, he reframed the problem as follows,

$$\min \int_0^{\infty} [B - U(C) + D(L)]dt \tag{71}$$

subject to the initial state of capital at,

$$K(0) = K_0$$

where B denote some arbitrary level of welfare bliss. The argument used was that since the integrand would have to minimize in each and every period, it will tend asymptotically to 0. However, as noted from our previous discussion and in Chiang (1992), just because the integrand tends to 0 as $t \to \infty$, does not guarantee that the functional would! However, since it uses implicit functions, we will assume that the functional does indeed converge as well.

To obtain the Euler Equation,

$$\begin{aligned} F_L &= -U_C Q_L + D_L & (72) \\ F_{L'} &= 0 & (73) \\ F_K &= -U_C Q_K & (74) \\ F_{K'} &= U_C & (75) \end{aligned}$$

These would allow us to analyse the solution using equation (16). First for the optimal path for labour,

$$\begin{aligned} F_L - \frac{dF_{L'}}{dt} &= 0 \\ \Rightarrow F_L &= 0 \\ \Rightarrow U_C Q_L &= D_L & (76) \end{aligned}$$

for all t which is the standard static labour choice, where the value of marginal product of labour must be equated to the marginal disutility of labour. For capital,

$$\begin{aligned} F_K - \frac{dF_{K'}}{dt} &= 0 \\ \Rightarrow -U_C Q_K &= U_{C,t} & (77) \\ \Rightarrow -Q_K &= \frac{U_{C,t}}{U_C} = \frac{\frac{dU_C}{dT}}{U_C} & (78) \end{aligned}$$

The last equality highlights the implications of the Euler equation. It says that the growth rate of the marginal utility of consumption must in each and every period be equal to the negative of the marginal product of capital. This growth path of consumption which is dependent on capital, would then identify the optimal growth path of labour.

Note in addition that the optimal growth of consumption together with labour then identifies implicitly or indirectly the optimal path of capital accumulation. Specifically, from equation (18), and our earlier discussion regarding special functional case when the functional is not dependent on t explicitly (20), we have then,

$$\begin{aligned} F - K'F_{K'} &= \text{constant} \\ \Rightarrow B - U(C) + D(L) - K'U_C &= \text{constant} \end{aligned} \tag{79}$$

To find the optimal path of capital accumulation path K', first note that as $t \to \infty$ we have $U(C) - D(L) \to B$. This in turn means that the minima can be achieve only as $U(C)$ tend towards its asymptotic value (recall that $U''(C) \leq 0$), and as $U(C)$ tends towards its asymptotic value, $U_C \to 0$ so that as $t \to \infty$, $B - U(C) + D(L) - K'U_C$ is 0. Therefore we have the optimal capital accumulation path as,

$$K' = \frac{B - U(C) + D(L)}{U_C} \tag{80}$$

and we get the famous *Ramsey Rule*. The rule says that the rate of capital accumulation have to set to equate the ratio between the short fall between the social welfare and the bliss level and the marginal utility of consumption. Note further that this idea relies on the production function through the marginal product's effect on the optimal path for consumption, equation (78).

It is common to also use *phase diagrams* to obtain qualitative information regarding the optimal paths. Since the model in its original form has three variables, consumption (or marginal utility from consumption), capital and investment, to depict these three variables and their paths would be complicated, we will simplify the analysis by assuming that labour input is constant for the remainder of the discussion, so that $Q \equiv Q(K)$, and similarly since L is no longer considered a state variable, $D(L)$ is eliminated from the functional so that it becomes in the aggregate $B - U(C)$ and being subjected to the same initial point constraint. Nonetheless, the Euler equation pertaining to consumption remains the same. To highlight the fact that we are concerned with the path of marginal utility and to see that the Euler Equation is in fact a differential equation, let $U_C = \mu$ so that we write equation (78) as,

$$\mu' + \mu Q_K = 0 \tag{81}$$

which is just a first order differential equation. From the relationship between consumption and capital accumulation we likewise get the second differential equation for K.

$$\begin{aligned} C(\mu) &= Q(K) - K' \\ \Rightarrow K' &= Q(K) - C(\mu) \end{aligned} \tag{82}$$

which is our second differential equation, and where $C(\mu)$ is derived from the fact that marginal utility of consumption is nondecreasing in C so that we can think of μ as being a function of C, so that inverting the function we get $C \equiv C(\mu)$.

Before we can depict the phase diagrams and their associated curves, note that we will maintain the previous assumptions that $U_{CC} \leq 0$ and similarly $Q_{KK} \leq 0$. Without loss of generality, let $\texttt{arg}\max \mu = C^*$ and $\max U(C) = B$. In constructing phase diagrams, we need first to know how $K' = 0$ and $\mu' = 0$ looks like or behaves. We need this two curves to separate the space of possible values of μ and K, and to see how they behave within those demarcated spaces. Since on the two curves, K' and μ' are zero, this means that K and μ are not changing along those lines. So that the intersection of the two curves tells us the *Intertemporal Equilibrium* or the *Steady State Equilibrium* for the system of equations. From the above two differential equations we obtain the following equations when $K' = 0$ and $\mu' = 0$,

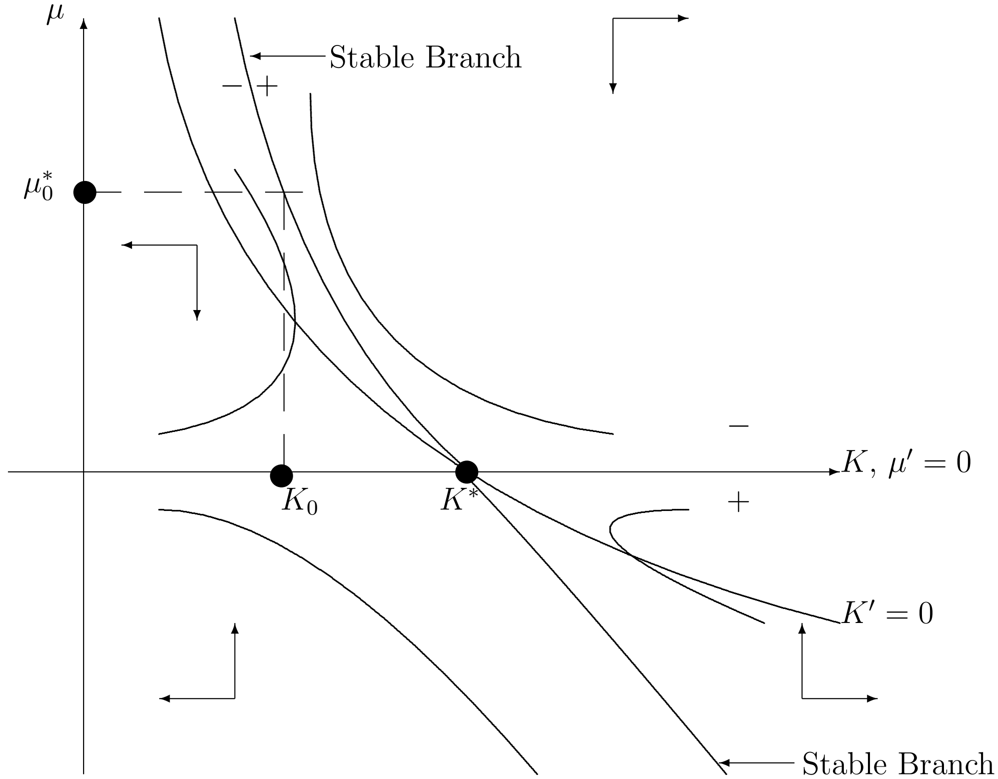
$$Q(K) = C(\mu) \tag{83}$$

$$\mu = 0 \tag{84}$$

The second equation obtains since Q is assumed to be strictly nondecreasing so that capital saturation never occurs. To determine the shape of equation (83), note that as capital increases, output likewise increases, which in turn implies that capital remaining for accumulation falls, so that consumption increases, and as consumption increases, marginal utility of consumption falls. In other words, capital (K) is a negative function of marginal utility of consumption μ, and equation (83) is a downward sloping curve. Since there is consumption saturation at B for utility, let capital associated with $\mu = 0$ at C^* be denoted as K^*. Similarly, equation (84) is just a straight flat line at $\mu = 0$. The two curves are depicted on figure 3.

To understand the "route" that the economy would have to take towards the steady state equilibrium, we have to next draw sketching bars (the arrows on figure 3) to understand how potential paths can move in the space of K and μ. First, we need to know how

Figure 3: Simplified Ramsey's Model



they behave using equation (81) and (82) respectively,

$$\frac{\partial \mu'}{\partial \mu} = -Q_K < 0 \tag{85}$$

$$\frac{\partial K'}{\partial K} = Q_K > 0 \tag{86}$$

This tell us how values of μ and K should behave in the four demarcated spaces divided by $\mu' = 0$ and $K' = 0$ through the sketching bars. For μ, using μ' as the demarcating line, the sign of the derivatives tells us that points to the north of $\mu' = 0$ must have values of μ falling as the streamlines moves towards $\mu' = 0$. While to the south they are rising. For K, values to the west of $K' = 0$, streamlines should see values of K falling, and to the east, they should be increasing. Thus there would exists a streamlines moving in a south-easterly direction towards the steady state equilibrium if the initial point has a low capital state, and in a north-westerly direction if the initial state were a high capital state. This steady state equilibrium is characterized as a saddle point at K^* where the $\mu' = 0$ and $K' = 0$ intersects. Note that to get onto the path towards the steady state equilibrium requires the social planner to find the correspondingly optimal level of marginal utility of consumption, which in figure 3 is μ_0^*, failing which the economy would not be able to reach its steady state of "bliss". For example, given K_0, if the social planner chooses a μ that is too low, which in turn is associated with too low levels of consumption, the economy would diverge towards to little capital accumulation. On the other hand, if the level of consumption were too high, associated with high levels of marginal utility of consumption, the economy would partake in to much capital accumulation. Only at μ_0^* is the economy driven towards the steady state. Put another way, the social planner has to follow a "specific rule" to have the economy get onto the path towards "bliss", as has been noted in our analysis. This result should not be surprising since if all paths taken moves the economy towards the steady state, there would not be a optimization problem to begin with.

References

Chiang, A. C. (1984): *Fundamental Methods of Mathematical Economics*. McGraw-Hill, New York: New York, 3 edn.

——— (1992): *Elements of Dynamic Optimization*. McGraw-Hill, New York: New York.

Ramsey, F. P. (1928): "A Mathematical Theory of Saving," *The Economic Journal*, pp. 543–559.

Taylor, D. (1989): "Stopping Inflation in the Dornbusch Model: Optimal Monetary Policies with Alternate Price-Adjustment Equations," *Journal of Macroeconomics*, pp. 199–216.