Advanced Macroeconomics, ECON 402
Infinite Horizon Model

We had previously in our initial development of dynamic optimization using the Calculus of Variations studied briefly Ramsey's model. As we have discussed in the Solow's model, it is the standard bearer, through which other growth models are compared. We will now revisit the Ramsey's model, but use the technique of Optimal Control Theory to solve it. In addition, we will now pay greater attention to the economic intuition, and compare the predictions with those of Solow's. We will also examine how the Government choices of taxation and deficit financing could alter capital accumulation within an economy, and consequently affect its growth rate.[1]

1 Infinite Horizon Model (Ramsey's Model Revisited): Command Economy

We will now build on our earlier discussion of Ramsey's (1928) model by allowing the labour force to grow. In addition, we will include the discount factor in the objective functional as opposed to the manner in which Ramsey (1928) had handled it. This version of the model is associated with Cass (1965) Nonetheless, despite this modifications, you will notice that the model retains much of the insight we obtained earlier. In addition, we will build on the analysis we developed in our examination of Solow's (1956) model, in terms of convergence rates in economic growth, and compare the two models explicitly.

1.1 Assumptions of the Model

We will build Ramsey's Model in the following manner,

1. The Economy is closed.

2. Let labour L_t grow at a rate n, and the effectiveness of labour be A_t and grows at

[1]This set of notes draws from Blanchard and Fischer (2000).

a rate g, which means

$$\frac{\dot{L}_t}{L_t} = n \tag{1}$$

$$\frac{\dot{A}_t}{A_t} = g \tag{2}$$

3. As before, let labour be supplied inelastically, in other words we do not consider unemployment in the model.

4. Output is a function of capital (K_t) and effective labour (A_tL_t). In other words,

$$Y_t = F(K_t, A_tL_t) \tag{3}$$

$$\tag{4}$$

Further, the production function is homogeneous of degree one as in Solow's (1956) model. Then we can write the production function in intensive form as follows,

$$\begin{aligned} \frac{Y_t}{A_tL_t} &= y_t \\ &= \frac{A_tL_tF(\frac{K_t}{A_tL_t}, 1)}{A_tL_t} \\ \Rightarrow y_t &= f(k_t) \end{aligned} \tag{5}$$

where we have denoted $f(k_t) = F(\frac{K_t}{A_tL_t}, 1)$.

5. As before, output is either consumed or invested,

$$Y_t = C_t + \dot{K}_t \tag{6}$$

$$\Rightarrow \frac{Y_t}{A_tL_t} = \frac{C_t}{A_tL_t} + \frac{\dot{K}_t}{A_tL_t}$$

$$\Rightarrow y_t = c_t + \dot{k}_t + nk_t \tag{7}$$

The last equality is derived from the fact that,

$$\begin{aligned} \dot{k}_t &= \frac{A_tL_t\dot{K}_t - K_tA_t\dot{L}_t - K_tL_t\dot{A}_t}{A_t^2L_t^2} \\ &= \frac{\dot{K}_t - k_tA_t\dot{L}_t - k_tL_t\dot{A}_t}{A_tL_t} \\ \Rightarrow \frac{\dot{K}_t}{A_tL_t} &= \dot{k}_t + (n+g)k_t \end{aligned} \tag{8}$$

Then k_t, c_t, y_t is capital, consumption, and output per unit of labour.

6. There is no depreciation of capital. Although this is a deviation from the Solow model, it is not a very significant one since all it needs is for you to think of the capital stock as the net capital stock.

7. As in Solow's (1956) model, $f(.)$ is increasing and strictly concave in k_t, and satisfies the Inada conditions, $f(0) = 0$, $f'(0) = \infty$, and $f'(\infty) = 0$

8. Initial capital is non-zero, $k_0 > 0$.

9. The utility of the infinitely lived representative individual or family unit is,

$$U_s = \int_s^{\infty} u(c_t)e^{-\rho(t-s)}dt \tag{9}$$

   where ρ is the discount factor and is strictly positive, and $u(c_t)$, the integrand is the instantaneous utility function of the agent/family unit, and is strictly increasing and concave.

1.2 Social Planner's Choice

We can state the Social Planner's problem as follows,

$$\max_{c_t} U_0 = \int_0^{\infty} u(c_t)e^{-\rho t}dt \tag{10}$$

$$\text{subject to} \quad f(k_t) = c_t + \dot{k}_t + (n+g)k_t \tag{11}$$

$$k_0 = A \quad A \text{ is given} \tag{12}$$

Put another way, it is the social planner's intention to maximize the representative agent or household's welfare. Note now that we will be using consumption c_t as the control variable, while k_t will be the state variable. In solving this problem, we are in effect solving it from the point of view of the social planner, and the equilibrium discovered is sometimes referred to as the *Command Optimum* equilibrium. We can then write the present value Hamiltonian as,

$$H = u(c_t)e^{-\rho t} + \lambda_t(f(k_t) - (n+g)k_t - c_t) \tag{13}$$

or alternatively since it is easier to work with the current value Hamiltonian, we can write,

$$H_c = u(c_t) + \gamma_t(f(k_t) - (n+g)k_t - c_t) \tag{14}$$

where as in our discussion of Optimal Control Theory, and the current value Hamiltonian, γ_t is the costate variable. Then using the Maximum Principle,

$$\begin{aligned}
\frac{\partial H_c}{\partial c_t} &= u_c(c_t) - \gamma_t = 0 & (15)\\
\dot{k}_t &= f(k_t) - c_t - (n+g)k_t & (16)\\
\dot{\gamma}_t &= \rho\gamma_t - \gamma_t(f'(k_t) - (n+g)) & (17)\\
& & (18)
\end{aligned}$$

and we have all the ingredients that we require to obtain a qualitative understanding of the model. First note that from equation (15) we have,

$$\begin{aligned}
u_c(c_t) &= \gamma_t \\
\Rightarrow u_{cc}(c_t)\dot{c}_t &= \dot{\gamma}_t & (19)
\end{aligned}$$

Then substituting equations (15) and (19) into equation (17) (which is the equation of motion for the costate variable) we have,

$$\begin{aligned}
\Rightarrow u_{cc}(c_t)\dot{c}_t &= u_c(c_t)(\rho + n + g - f'(k_t)) \\
\Rightarrow \frac{u_{cc}(c_t)\dot{c}_t}{u_c(c_t)} &= \rho + n + g - f'(k_t) & (20)
\end{aligned}$$

which is, as you may recognize, the Ramsey's Rule (or sometimes referred to as the *Keynes-Ramsey Rule*). If you do not see it, just realize that whereas we had stated Ramsey's Rule in terms of $\dot{K}$ in our previous discussion, where K is the state variable, here it is stated in terms of the control variable c_t.

At this juncture, if you look hard, you may recognize that the coefficient to $\dot{c}_t$ looks like a elasticity measure of sorts, or what you may recognize in our discussion of the Calculus of variations when we were depicting the phase diagram. You would be right on both counts, in that it is the elasticity of the marginal utility with respect to consumption, and it describes the curvature of the utility function. In fact there is a better interpretation of the ratio $\frac{u_{cc}(c_t)}{u_c(c_t)}$ is to notice that it is related to the instantaneous elasticity of substitution. Consider the following, the elasticity of substitution of consumption across two period is just,

$$\begin{aligned}\sigma(c_t) &= -\frac{d(c_s/c_t)}{d(u'(c_s)/u'(c_t))}\frac{u'(c_s)/u'(c_t)}{c_s/c_t} \\ \Rightarrow \lim_{s\to t}\sigma(c_t) &= -\lim_{s\to t}\frac{d(c_s/c_t)}{d(u'(c_s)/u'(c_t))}\frac{u'(c_s)/u'(c_t)}{c_s/c_t} \\ &= -\lim_{s\to t}\frac{u'(c_t)/c_t}{du'(c_s)/dc_s}\frac{u'(c_s)/c_s}{u'(c_t)/c_t} \\ &= -\frac{u_c(c_t)}{u_{cc}(c_t)c_t}\end{aligned}$$

which reveals that on the limit, as $s \to t$ we get the instantaneous elasticity of substitution, which is the negative of the inverse of the coefficient to $\dot{c}_t$ of our Keynes-Ramsey rule, so that we can write the Keynes-Ramsey rule as,

$$\frac{\dot{c}_t}{c_t} = \sigma(c_t)[f'(k_t) - \rho - (n+g)]$$

This condition is essentially the continuous time version of the standard efficiency requirement since your first year, that at equilibrium, marginal rate of substitution should be equal to the marginal rate of transformation. You should read Blanchard and Fischer (2000) if you are interested to know a more intuitive, but technical interpretation of the Keynes-Ramsey rule in discrete time.

We have solved much of the problem, with the exception of verifying that the transversality condition are met. Noting that since this is the infinite horizon problem, our transversality conditions based on our earlier discussion of Optimal Control Theory are,

$$\lim_{t\to\infty} H = 0 \tag{21}$$

$$\lim_{t\to\infty} \lambda_t = 0 \tag{22}$$

To see that these conditions are met, first note that by the assumptions we have made regarding the utility function, c_t^* must be an interior solution, in other words, be finite. This then means that the first term of the Hamiltonian in equation (13) will tend towards zero as t tends towards ∞. This then leaves the second term. To understand how it behaves at $t \to \infty$, we have to examine the second transversality condition. From the maximum principle, equation (15), we have

$$\begin{aligned}\gamma_t^* &= u_c(c_t^*) \\ \Rightarrow \lambda_t^* &= u_c(c_t^*)e^{-\rho t} \\ \Rightarrow \lim_{t\to\infty} \lambda t^* &= 0\end{aligned}$$

where the last equality follows since by assumption $u(.)$ is strictly increasing and concave. Therefore, the final term in the Hamiltonian of equation (13) will tend to zero. Thus all the transversality conditions are met.

We are now ready to examine the qualitative aspects of the equilibrium path of such an economy using a Phase Diagram. To do so, we would need the differential equations for k_t and c_t which are equations (16) and the Keynes-Ramsey rule of (20) respectively. We know that at steady state, $\dot{k}_t = 0$, which gives us,

$$\begin{aligned} c_t &= f(k_t) - (n+g)k_t \\ \Rightarrow \frac{\partial c_t}{\partial k_t} &= f'(k_t) - (n+g) \\ \Rightarrow \frac{\partial c_t^2}{\partial^2 k_t} &= f''(k_t) < 0 \end{aligned}$$

which says that output per unit of effective labour is divided between consumption and the maintenance of capital in lieu of population growth and growth in effectiveness of labour. Further, the locus of $\dot{k}_t$ is concave. For when consumption is in steady state, we have $\dot{c}_t = 0$, from which we derive a *Modified Goldern Rule* since it is now augmented by the discount factor,

$$f'(k_t) = \rho + n + g$$

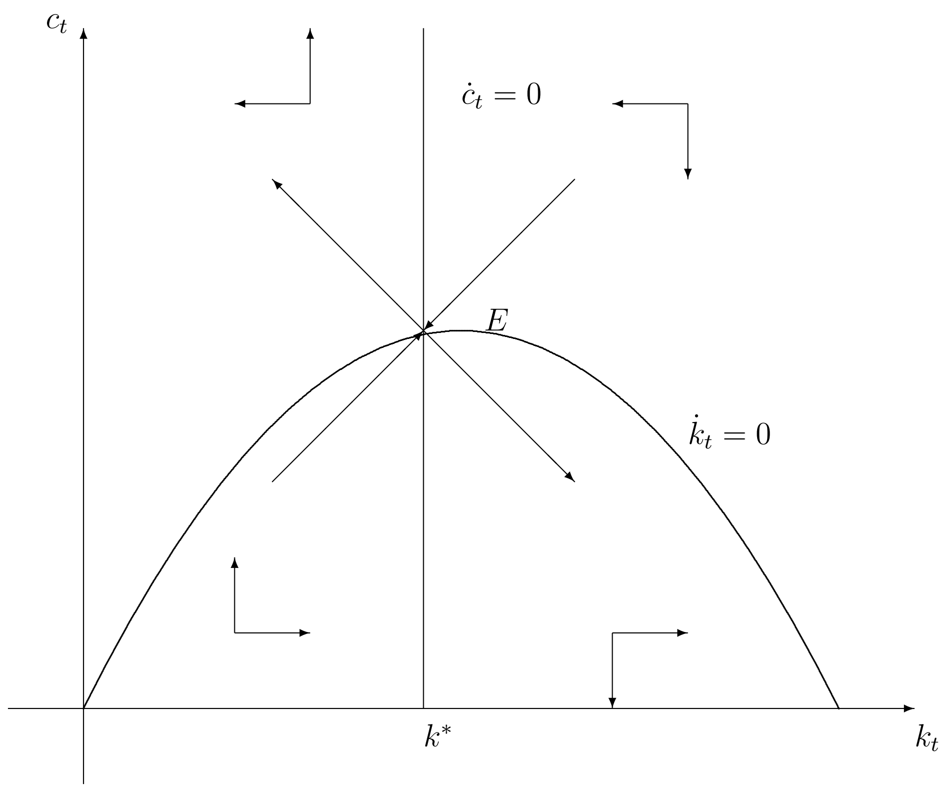
You should recall from our discussion previously of Solow's (1956), the marginal productivity of capital is set equal to the maintenance of capital. However, based on the Keynes-Ramsey Rule, the discount factor is included since we have to contend with the fact that present consumption typically weighs heavier in our minds. Consequently, the increase in the right hand side of the equation of the *Golden Rule*, gives that the production function is increasing and concave, implies that k_t would have to be lower. This *Modified Golden Rule* has powerful implications. Since the level of capital determines the productivity of capital, and consequently real interest rate (through the return of capital in production), it means that our impatience manifested in the discount factor, population growth, and growth in effectiveness will determine our rate of growth as an economy. At this juncture, you should notice that this was not what we observed in Ramsey's (1928) original model since he had not included the discount factor. This version of the model we are currently discussing is associated with Cass (1965). Finally, note that at this locus of points associated with $\dot{c}_t = 0$, we have a constant k_t^* so that it is a vertical line. Both of the loci are depicted below together with the sketching bars. To define the sketching bars, we have to determine how the deviations outside of the loci would move, which as before, we do using the differential equations for $\dot{c}_t$ and $\dot{k}_t$.

$$\frac{\partial \dot{k}_t}{\partial c_t} = -1 < 0 \tag{23}$$

$$\frac{\partial \dot{c}_t}{\partial k_t} = \sigma(c_t)c_t f''(k_t) < 0 \tag{24}$$

which says that for points to the north of the $\dot{k}_t = 0$, where consumption is increasing in relation to points on the locus, k_t must be decreasing, or in other words moving in a westward direction, while below it, in a eastward direction. Similarly, for points to the left of the $\dot{c}_t = 0$ locus, k_t is falling in relation to points on the locus, so that c_t must be increasing, or moving in a northward direction, and those points to the right in a southward direction. All of which are depicted on figure 1.

Figure 1: Dynamics of Capital and Consumption



It clear that it is only at point E do we get to a steady state equilibrium, and it is only there that both the conditions set out by the differential equations, i.e. that determining consumption and the Keynes-Ramsey rule, as well as the transversality conditions are met. And the stream lines that point towards point E are the only ways to get to that *saddle point*[2].

2 Decentralized Economy

Since the individual is an element in the economy, and there are firms, unlike in Solow's (1956) model, it is worth examining if the individual making her own choices could do better or worse or no different than the social planner's choice. We will make the following assumptions in addition to what was made earlier, most of which are restatements, while others are meant to clarify the necessary framework within a complete decentralized economy.

2.1 Assumptions of the Model

1. The Economy is closed.

2. The choices to be made by the individuals or families are how much labour and capital to supply to the firms (from the supply of which they obtain wages w_t, and rent r_t.), and how much to save or consume. Each individual unit can either save by accumulating capital or lend to other units in the economy.

3. We will assume that this units are indifferent between accumulating capital or lending, so that the interest rate on debt and rental rate of capital are equal. These choices are made based on the lifetime (infinite horizon) utility function of the family unit.

4. The individual unit then faces the following problem precisely,

$$\max_{c_t} U_s = \int_0^\infty u(c_t)e^{-\rho(t-s)}dt \tag{25}$$

$$\text{subject to} \quad c_t + \dot{h}_t + (n+g)h_t = w_t + r_t h_t \quad \forall t \tag{26}$$

$$k_0 = A \quad A \text{ is given.} \tag{27}$$

[2] Both the origin, and the intersection between the $\dot{k}_t$ locus and the horizontal axis are equilibrium points as well, thought they do not satisfy all the requirements of the model. For details read Blanchard and Fischer (2000)

where $h_t \equiv k_t - d_t$, which is just family unit wealth. It is dependent of capital in their possession less the debt they possess d_t.

To understand the derivation of the budget constraint, note that for each unit of effective labour, consumption is net of her effective wages, net wealth from her investments or savings, and her investment rate. In other words,

$$\begin{aligned}
C_t &= A_tL_tw_t + (K_t - D_t)r_t - (\dot{K}_t - \dot{D}_t) \\
C_t/A_tL_t &= w_t + (K_t - D_t)r_t/A_tL_t - (\dot{K}_t - \dot{D}_t)/A_tL_t \\
c_t &= w_t + (k_t - d_t)r_t - (\dot{K}_t - \dot{D}_t)/A_tL_t \\
c_t &= w_t + h_tr_t - \dot{h}_t - (n+g)h_t
\end{aligned}$$

since,

$$\begin{aligned}
\frac{\partial (K_t - D_t)/A_tL_t}{\partial t} &= \frac{(\dot{K}_t - \dot{D}_t)L_t - (A_t\dot{L}_t + L_t\dot{A}_t)(K_t - D_t)}{A_t^2L_t^2} \\
\dot{h}_t &= \frac{(\dot{K}_t - \dot{D}_t)}{A_tL_t} - (n+g)\frac{K_t - D_t}{A_tL_t} \\
\dot{h}_t &= \frac{(\dot{K}_t - \dot{D}_t)}{A_tL_t} - (n+g)h_t
\end{aligned}$$

To draw links to the model in your text, this model here essentially assumes that the rate of growth of the household mirrors that of the entire economy.

5. To ensure this problem has its parallel in the centralized command optimization problem, as before, both capital and labour are supplied to the firms inelastically. This means then that each unit considers only how much to consume and save in each period.

6. Likewise, each identical firm has the same technology, and they choose how to rent in terms of capital and effective labour. The firms are assume to exist in a competitive market(s) and take prices, real wages and rents as given.

7. Since these competitive firms are profit maximizers, it is common knowledge to you now that the first order conditions to profit maximization are,

$$f'(k_t) = r_t \tag{28}$$

$$f(k_t) - k_tf'(k_t) = w_t \tag{29}$$

To see this, note that the profit function can be written as,

$$\begin{aligned}
\Pi(K_t, A_tL_t) &= F(K_t, A_tL_t) - A_tL_tw_t - K_tr_t \\
\Rightarrow \Pi(K_t, A_tL_t) &= A_tL_tF(K_t/A_tL_t, 1) - A_tL_tw_t - K_tr_t \\
\Rightarrow \frac{\partial \Pi(K_t, A_tL_t)}{\partial K_t} &= F'(K_t, A_tL_t) - r_t \\
&= A_tL_tF'(K_t/A_tL_t, 1)(1/A_tL_t) - r_t \\
&= f'(k_t) - r_t = 0 \\
\& \quad \Rightarrow \frac{\partial \Pi(K_t, A_tL_t)}{\partial A_tL_t} &= F(K_t/A_tL_t, 1) - A_tL_tF'(K_t/A_tL_t, 1)K_t/(A_tL_t)^2 - w_t \\
&= f(k_t) - k_tf(k_t) - w_t = 0
\end{aligned}$$

where r_t is the real interest rate, and w_t is the real wage rate for effective labour.

8. Both of these groups have *Perfect Foresight* so that they can anticipate all current and future w_t and r_t.

9. *No-Ponzi-Game Condition* is assumed so that individuals and families cannot have exploding debt. This condition is stated as,

$$\lim_{t\to\infty} h_t e^{-\int_0^\infty (r_s - n - g)ds} \geq 0 \tag{30}$$

Essentially, from the budget constraint (26), if the individual funds an ever increasing consumption, wealth h_t will eventually become negative, and this indebtedness will be growing at a rate of $r_t - n - g$. Then the No-Ponzi-Game condition says that family debt cannot asymptotically grow at a faster rate than the interest rate.

Note that as long as the marginal utility of consumption is greater than zero, neither would individual choose to accumulate wealth forever, so that the condition (30) will be binding, and hold with equality.

To see the implications of the condition, note first that $h_t = h_0 e^{-\int_0^t (r_s - n - g)ds}$, which implies that $\dot{h}_t = -h_0 e^{-\int_0^t (r_s - n - g)ds}(r_t - n - g)$ (by Liebniz Rule), so that $\dot{h}_t = -h_t(r_t - n - g)$, which is the net rate of return from wealth. To understand the implications of the No-Ponzi condition, we can examine an arbitrary sequence of current value of consumption constraint from 0 to T.

$$\begin{aligned}
\int_0^T c_t e^{\int_t^T [r_s-n-g]ds} dt + \int_0^T [\dot{h}_t - (r_t-n-g)h_t] e^{\int_t^T [r_s-n-g]ds} dt &= \int_0^T w_t e^{\int_t^T [r_s-n-g]ds} dt \\
\Rightarrow \int_0^T c_t e^{\int_t^T [r_s-n-g]ds} dt + h_t e^{\int_t^T [r_s-n-g]ds} \Big|_0^T &= \int_0^T w_t e^{\int_t^T [r_s-n-g]ds} dt \\
\Rightarrow \int_0^T c_t e^{\int_t^T [r_s-n-g]ds} dt + h_T - h_0 e^{\int_0^T [r_T-n-g]ds} &= \int_0^T w_t e^{\int_t^T [r_s-n-g]ds} dt
\end{aligned}$$

Next, discounting the constraint to its present value by $e^{-\int_0^T [r_s-n-g]ds}$ we have

$$\begin{aligned}
\int_0^T c_t e^{-\int_0^t [r_s-n-g]ds} dt + h_T e^{-\int_0^t [r_s-n-g]ds} - h_0 &= \int_0^T w_t e^{\int_0^t [r_s-n-g]ds} dt \\
\Rightarrow \lim_{T\to\infty} \int_0^T c_t e^{-\int_0^t [r_s-n-g]ds} dt + \lim_{T\to\infty} h_T e^{-\int_0^t [r_s-n-g]ds} - h_0 &= \lim_{T\to\infty} \int_0^T w_t e^{\int_0^t [r_s-n-g]ds} dt \\
\Rightarrow \int_0^\infty c_t e^{-\int_0^t [r_s-n-g]ds} dt &= \int_0^\infty w_t e^{\int_0^t [r_s-n-g]ds} dt + h_0
\end{aligned}$$

where the last condition implies that the present value of lifetime consumption must be equal to accumulated earnings, and wealth, as we had wanted. It is the No-Ponzi condition that allows us to use the constraint (26) as opposed to this inter-temporal constraint.

2.2 Decentralized Equilibrium

To solve for the individual unit's problem, the Hamiltonian can be written as,

$$\begin{aligned}
H &= u(c_t)e^{\rho t} + \lambda_t[w_t + (r_t-n-g)h_t - c_t] && (31) \\
\Rightarrow H_c &= u(c_t) + \gamma_t[w_t + (r_t-n-g)h_t - c_t] && (32)
\end{aligned}$$

Then again, finding the Maximum Principle conditions yields (where here we have c_t as the control variable, and h_t as the state variable)

$$\begin{aligned}
\frac{\partial H_c}{\partial c_t} &= u_c(c_t) - \gamma_t = 0 && (33) \\
\dot{h}_t &= w_t + (r_t-n-g)h_t - c_t && (34) \\
\dot{\gamma}_t &= \rho\gamma_t - \gamma_t(r_t-n-g) && (35)
\end{aligned}$$

First note that,

$$\begin{aligned}
\dot{\gamma}_t &= u_{cc}(c_t)\dot{c}_t \\
\Rightarrow u_{cc}(c_t)\dot{c}_t &= u_c(c_t)(\rho - r_t - n - g) \\
\Rightarrow \frac{u_{cc}(c_t)\dot{c}_t}{u_c(c_t)} &= \rho + n + g - f'(k_t)
\end{aligned}$$

which is exactly the *Keynes-Ramsey Rule* we obtained from the command economy problem. In other words, the dynamics associated with this decentralized economy is as in our command economy.

At this juncture, it is important for you to notice the primary importance that *expectations* take in this model. All decisions are made based on knowledge of the paths that wages and rental/interest rates would take. The mechanism can be characterized in general as follows: First, interest rates determine the *marginal propensity to consume* out of the individual unit's wealth, and in turn the value of wealth through the individual's lifetime income. Secondly, the expectations on wages will determine consumption through the lifetime source of income it generates. This expectations together determine consumption and saving choices. Consequently, capital accumulation is determined and the sequence of factor prices.

Finally, note that since this economy is competitive, has no externalities, and that all agents are homogeneous, the *First Welfare Theorem* holds, and this economy is *Pareto Efficient*, and by extension, the command economy is likewise.

3 Government in the Decentralized Economy

We can build on the model by including considerations for the government. We will make the following assumptions.

1. Government expenditures are exogenous. These expenditures are financed by either taxation or borrowings.

2. Denote government's per capita demand for resources is denoted as m_t. This is funded by a per capita lump-sum tax of τ_t. Let $\tau_t = m_t$ so that the government's budget is always balanced.

3. The household's budget constraint is now,

$$c_t + \dot{h}_t \;=\; w_t + (r_t - n - g)h_t - \tau_t$$

As we have found, we can rewrite the constraint in inter-temporal form as,

$$\int_0^\infty c_t e^{-\int_0^t [r_s - n - g]ds} dt \;=\; \int_0^\infty w_t e^{\int_0^t [r_s - n - g]ds} dt + h_0 - \int_0^\infty \tau_t e^{\int_0^t [r_s - n - g]ds} dt \quad (36)$$

This then illustrates how government spending, since the budget must be balanced, could affect the time paths of wages and interest/rental rates, through the consequent effect on individual choices.

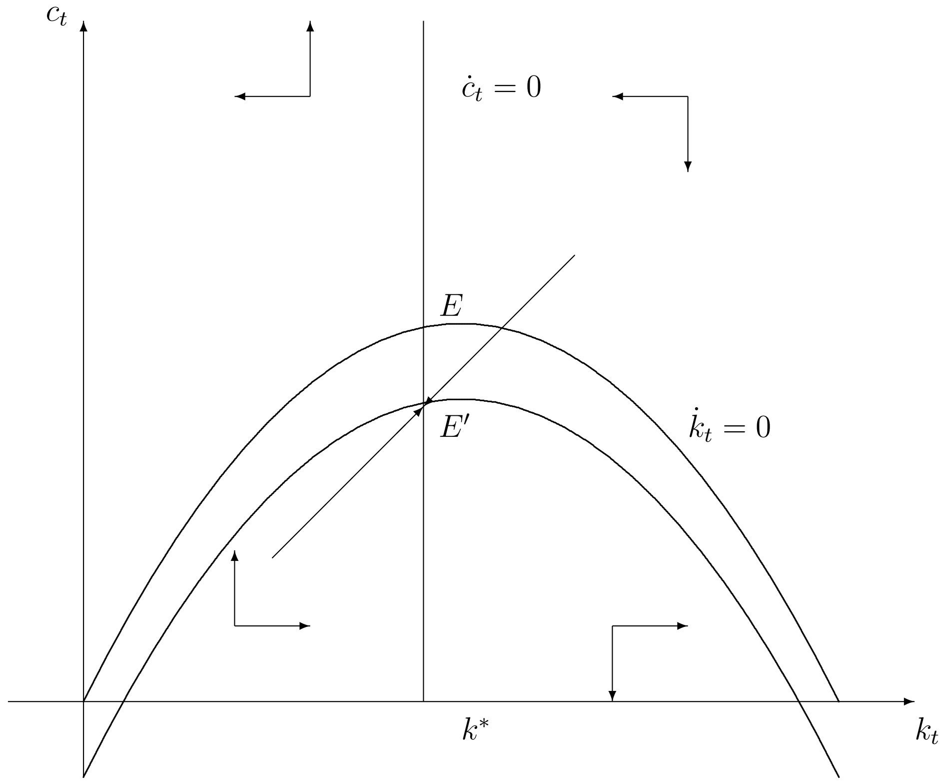
3.1 Effect of Government Spending

We can now examine the implications of government spending. Given that there is a complete parallel between the centralized and decentralized economy in this model, all we need then is to translate these new assumptions into that framework. To simplify the analysis, let the demand on resources by the government be constantly growing at a rate of $m_t = m$ for all t. This then means that the per capita stock of capital is,

$$f(k_t) \;=\; c_t + \dot{k}_t + (n + g + m)k_t \quad (37)$$

In terms of the decentralized economy, what this setup does is to reduce the wealth of the individual unit, or in other words, to reduce $h_t = k_t - d_t - m_t$. The consequent effect is to instantaneously reduce private consumption c_t should the economy be already in steady state equilibrium. This is depicted in figure (2). In other words, government spending completely crowds out private consumption without any effect on the capital stock. The change in steady is from E to E'.

Figure 2: Effects of Public Spendng



3.2 Debt Financing

Of course the government can finance its expenditures using debt as opposed to taxation, or both. Let government debt be denoted by p_t, and the dynamic budget constraint of the government can written as,

$$\dot{p}_t + (n+g)p_t = m_t - \tau_t + r_t p_t$$

where the rate of return on government debt must be equal to private debt, r_t. We can get the inter-temporal budget constraint by performing the same operation as we did for the individual family unit in the decentralized economy, as well as applying the No-Ponzi condition to the government. The inter-temporal constraint is,

$$p_0 + \int_0^\infty m_t e^{\int_0^t (r_s - n - g)ds} dt = \int_0^\infty \tau_t e^{\int_0^t (r_s - n - g)ds} dt \tag{38}$$

(**Show yourself how to get it the inter-temporal constraint.**) The above says then that the present value of government taxes must be equal to the sum of the initial government debt and the present value of government spending. If in the initial period, the government has positive debt, $p_0 > 0$, then based on the inter-temporal budget constraint, the government can maintain the initial value of per capita debt forever by running a budget surplus forever.

The existence of government debt will likewise affect the individual economic units budget constraint by having $h_t = k_t - d_t + p_t$. The structure of the budget constraint remains unchanged. The inter-temporal version of the budget constraint likewise has the same structure.

What is important to note is that based on the government's budget constraint, there is no necessity for the government to maintain a perpetual balanced budget. In other words, it could reduce current taxes, and finance its expenditures through public debt, and raise taxes in some future to repay those debt and interest. This gives you the perennial complaint that the government can pass on current burden onto future generations. It is perhaps interesting to ask now whether the timing of taxation has any real effects on the economy. It turns out that there is no effect at all! To see this, substitute the government's inter-temporal budget constraint into the individual's inter-temporal budget constraint. What you should obtain is the individual unit's budget constraint under the balanced budget equation of (36). In other words, it does not matter how expenditures are financed, but that public spending crowds out private consumption in this model. Put another way, debt financing and lump-sum taxation has no distortionary effects on capital stock and generally for capital accumulation. This conclusion is interesting in that it says that the size of the national debt and deficit finance is of no consequence in the long run, so long as the No-Ponzi condition is met! Nonetheless, this result is rather strong!

3.3 Distortionary Taxes

Suppose the taxation scheme used by the government is on the return to capital, at a rate of ξ_t, so that it alters the effective rate of return on capital to $(1-\xi_t)r_t$, the after-tax return. These tax receipts without loss of generality are transfered back to the individuals on a per capita basis of z_t which is equal to the receipts. Then the individual unit's dynamic budget constraint is now,

$$c_t + \dot{h}_t + (n+g)h_t = w_t + (1-\xi_t)r_t h_t + z_t \tag{39}$$

It is easy to see that this alters the first condition of the Maximum Principle so that the *Keynes Ramsey Rule* is now,

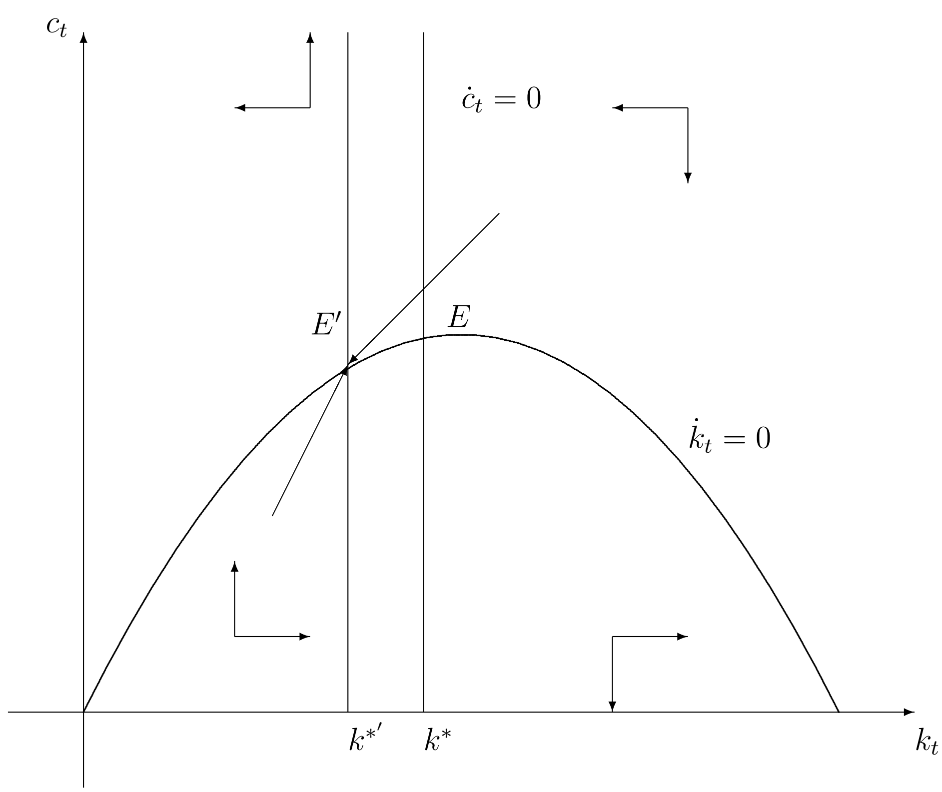
$$\frac{u_{cc}(c_t)\dot{c}_t}{u_c(c_t)} = \rho + n + g - (1-\xi_t)r_t \tag{40}$$

This then means that the optimal capital stock is now altered. To see this, we know that in steady state, $\dot{c}_t = 0$, and we know that $f(.)$ and $f'(.)$ are monotonic functions, so that a inverse exists.

$$\begin{aligned} f'(k_t^*) &= \frac{\rho + n + g}{1-\xi_t} \\ \Rightarrow k_t^* &= f'^{-1}\left(\frac{\rho + n + g}{1-\xi_t}\right) \end{aligned}$$

In other words, the steady state capital stock is now lower for a positive tax rate. Consequently, the steady state consumption level is also lowered. Of course, if the government subsidized capital through lump sum taxation, the opposing outcome would be derived. Diagrammatically,

Figure 3: Effects of Distortionary Taxes



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