Advanced Macroeconomics, ECON 402
Overlapping Generations Model

We have now covered the original Ramsey's (1928) model which used the concepts associated with the Calculus of Variations, and it variation using Optimal Control Theory developed by Cass (1965). These models are all variations of Neoclassical Growth Models. The key difference between these Neoclassical Approach and that of Keynes' is that they incorporated microeconomic concerns. We will now add to these sets of models the Overlapping generations model developed by Samuelson (1958) and Diamond (1965). [1]

1 The Overlapping Generations Model

The distinguishing feature of Ramsey's (1928) and Cass's (1965) models are that the agents are infinitely lived. A relevant question is whether if we remove that assumption so that generations overlapped, and consequently had to trade with one another, and if the horizon of the economy were allowed to persist ad infinitum, how the steady state equilibrium might or might not differ? To examine this possibility, we will now develop the Overlapping Generations model.

The Overlapping Generations Model continues to be widely used today, and one of the reasons is that it is capable of analyzing the implications of the life-cycle behavior of agents. This is possible because individuals are driven by their need to save during their active work live to finance their retirement consumption and the welfare of future generations, and this interplay between the incentives of the current generation and government policy provides the basis for a more complete examination of "true" behavior within an economy. In fact, this model allows for the possibility that the decentralized competitive equilibrium be different from that of the social planner's choice. In fact, this competitive equilibrium need not even be Pareto Efficient, which provides a point of departure from that generated by Ramsey (1928) and Cass (1965).

[1] This set of notes draws from Blanchard and Fischer (2000).

1.1 The Assumptions

We will first examine the public sector free equilibrium of the decentralized economy. As usual, to develop the model we will state the assumptions.

1. Individuals live for two periods, so that in any period the economy has two cohorts interacting with each other.

2. Unlike the previous models, time is discrete now.

3. The market economy is composed of both individual consumers and firms.

4. An individual born at period t, consumes $c_{1,t}$ in period t, and consumes $c_{2,t+1}$ in period $t+1$, so that her lifetime utility is,

$$u(c_{1,t}) + (1+\rho)^{-1}u(c_{2,t+1}) \tag{1}$$

   where $\theta \geq 0$, and $u'(.) > 0$, and $u''(.) < 0$.

5. Individuals work only in the first period of their life, supplying labour inelastically at a wage rate of w_t. Part of this income is saved and consumed in the second period during their retirement.

6. Savings of individuals in period t generates the capital stock of period $t+1$ which is used in combination of labour supplied in period $t+1$ to produce the output of period $t+1$.

7. The number of individuals born at time t enters the labour force at time t, and this number is L_t. Labour grows at a rate n, that is $L_t = L_0(1+n)^t$.

8. Effectiveness of labour grows at rate g, so that $A_t = A_0(1+g)^t$.

9. Firms as before act competitively, and the production technology has constant returns to scale as before. That is $Y_t = F(K_t, A_tL_t)$. Then as before, output per effective labour is $Y_t/A_tL_t = F(K_t/A_tL_t, 1) = f(k_t)$. Further, the production function is increasing and strictly concave, and satisfies the Inada Conditions. Each firm by virtue of the competitive nature, takes the wage rate w_t, and rental rate r_t as given.

1.2 The Decentralized Equilibrium

In this initial equilibrium, we will not consider bequest, in other words, individuals do not care for their future generations. This then implies that each generation completely consumes all their savings before they exit their life. The maximization problem facing an individual born at time t is,

$$\begin{aligned} & \max_{c_{1,t},c_{2,t+1}} u(c_{1,t}) + \frac{u(c_{2,t+1})}{1+\rho} \qquad (2) \\ \text{subject to} \quad & c_{1,t} + s_t = w_t \\ & c_{2,t+1} = (1+r_{t+1})s_t \end{aligned}$$

The two constraints can be reduced to a single one,

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = w_t \qquad (3)$$

This problem can easily be solved by setting up the Lagrangian or by simply substituting the constraints into the objective function. We will take the more systematic route, and write the Lagrangian,

$$\mathcal{L} = u(c_{1,t}) + \frac{u(c_{2,t+1})}{1+\rho} + \lambda\left(w_t - c_{1,t} - \frac{c_{2,t+1}}{1+r_{t+1}}\right) \qquad (4)$$

Therefore the first order conditions are,

$$\frac{\partial \mathcal{L}}{\partial c_{1,t}} = u'(c_{1,t}) - \lambda = 0 \qquad (5)$$

$$\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} = \frac{u'(c_{2,t+1})}{1+\rho} - \frac{\lambda}{1+r_{t+1}} = 0 \qquad (6)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w_t - c_{1,t} - \frac{c_{2,t+1}}{1+r_{t+1}} = 0 \qquad (7)$$

From the first two first order conditions, we have,

$$u'(c_{1,t}) = \frac{u'(c_{2,t+1})(1+r_{t+1})}{1+\theta} \qquad (8)$$

This equation, through the two constraints of the individuals problem implicitly defines the effect that wage rate and rental rate has on the savings rate, $s_t = s(w_t, r_{t+1})$. The effect that each element has on the savings rate specifically is,

$$u''(c_{1,t})(1-s_w) - \frac{u''(c_{2,t+1})(1+r_{t+1})^2 s_w}{1+\rho} = 0 \qquad (9)$$

$$\Rightarrow \quad 0 < s_w = \frac{u''(c_{1,t})}{u''(c_{1,t}) + \frac{u''(c_{2,t+1})(1+r_{t+1})^2}{1+\rho}} < 1 \qquad (10)$$

and

$$
\begin{aligned}
-u''(c_{1,t})s_r + \frac{u''(c_{2,t+1})(1+r_{t+1})(s_t + (1+r_{t+1})s_r) + u'(c_{2,t+1})}{1+\rho} = 0 \\
\Rightarrow \quad s_r = \frac{u''(c_{2,t+1})(1+r_{t+1})s_t + u'(c_{2,t+1})}{(u''(c_{1,t})(1+\rho) - u''(c_{2,t+1})(1+r_{t+1})^2)} \lesseqqgtr 0
\end{aligned}
$$

Based on the above comparative statics, savings is an increasing function of wages, and neither is it surprising that the change in savings rate is less than 1. On the other hand, and increase in interest rate has an ambiguous effect on savings. This ambiguity in this two period model is created by the opposing effects if income and substitution effects. When interest rates increase, the price of second period consumption falls, thereby providing the incentive for the individual to substitute away from future consumption toward present period consumption, in other words reducing savings. On the other hand, the increase in interest rate has an income effect of raising lifetime income and wealth, so that the incentive is to increase savings so that future consumption increases as well.

As in our discussion of in the Infinite Horizon model, firms will pay wages equal to the marginal product of effective labour, and rental rate equal to the marginal product of capital. In other words,

$$
\begin{aligned}
f(k_t) - k_t f'(k_t) &= w_t \\
f'(k_t) &= r_t
\end{aligned}
$$

Insofar as firms being willing to pay these wages, and rental rates, this necessarily implies that the factor market is in equilibrium as well.

To find the goods market equilibrium, note first that the demand and supply of goods must equate in each period, or that savings must equate with investments. This translates to,

$$
K_{t+1} - K_t = A_t L_t s(w_t, r_{t+1}) - K_t
$$

where the left hand side of the equation is the net increase in stock of capital, in other words investment, while the right hand side is the net savings given by the savings of the young cohort less the withdrawal by the old cohort of their savings. In intensive form, the above equation can be written as,

$$\begin{aligned}
K_{t+1} &= A_t L_t s(w_t, r_{t+1}) \\
\Rightarrow K_{t+1}/A_{t+1}L_{t+1} &= s(w_t, r_{t+1})A_t L_t/A_{t+1}L_{t+1} \\
\Rightarrow k_{t+1} &= s(w_t, r_{t+1})1/(1+n)(1+g) \\
\Rightarrow k_{t+1}(1+n)(1+g) &= s(w_t, r_{t+1}) \qquad (11)
\end{aligned}$$

Together, equations (11) and (11) describes the dynamic behavior of the capital stock, and consequently the growth of the economy. We can thus write the conditions as,

$$k_{t+1} = \frac{s(f(k_t) - k_t f'(k_t), f'(k_{t+1})}{(1+n)(1+g))} \qquad (12)$$

which then describes the savings locus. This locus is dependent on the effect that k_t has on k_{t+1}.

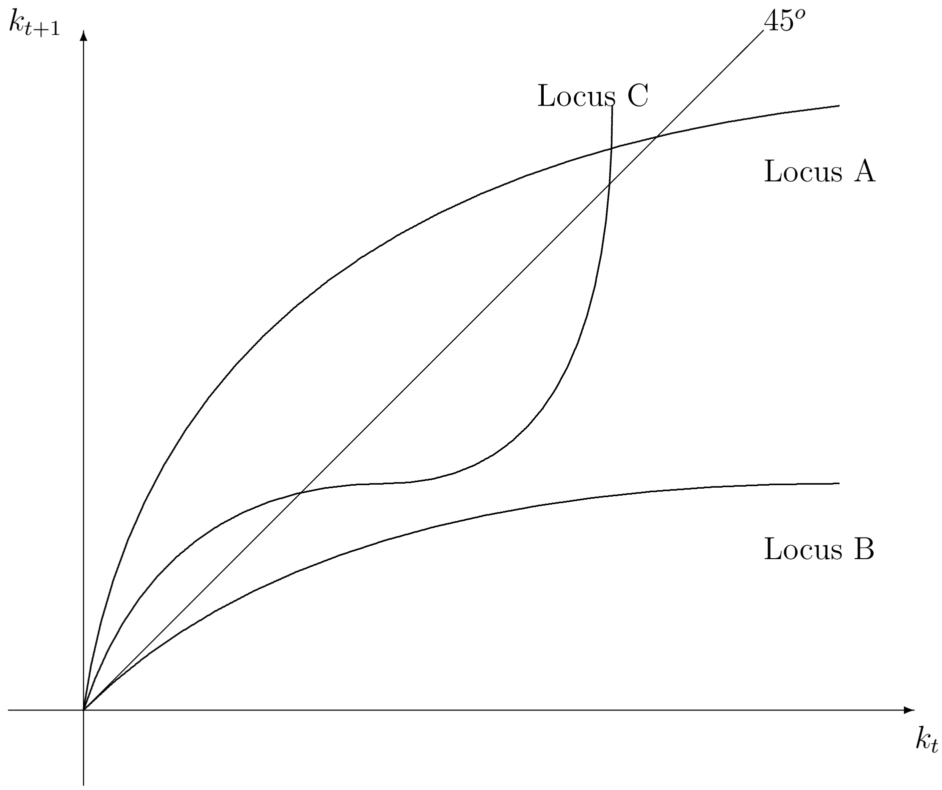
$$\begin{aligned}
\frac{dk_{t+1}}{dk_t} &= \frac{s_w[f'(k_t) - f'(k_t) - k_t f''(k_t)] + s_r f''(k_{t+1})\frac{dk_{t+1}}{dk_t}}{(1+n)(1+g)} \\
\Rightarrow \frac{dk_{t+1}}{dk_t} &= \frac{-s_w k_t f''(k_t)}{(1+n)(1+g) - s_r f''(k_{t+1})} \lesseqqgtr 0 \qquad (13)
\end{aligned}$$

Although the numerator is positive, there remains ambiguity regarding the evolution of capital stock. The derivative is positive if the effect that rental rate has on savings is positive, $s_r > 0$. The implications are as follows: we know that in steady state, $k_t = k_{t+1}$ so that savings is growing at a constant rate. Diagrammatically, this would a 45^o line such as in figure 1. Then a savings locus that crosses the 45^o line would be a steady state equilibrium.

The first question we would like to answer is whether under the current circumstances, if a steady state is guaranteed? Given the ambiguity in sign of the derivative of equation (13), what it means however is that the general form of our utility and production functions do not provide sufficient restrictions to define a shape for the savings locus.

A second question is then given the existence of a steady state with a positive capital stock, whether the equilibrium is indeed stable? The question of stability relates to whether when an economy reaches a equilibrium steady state, if

Figure 1: Savings Steady States



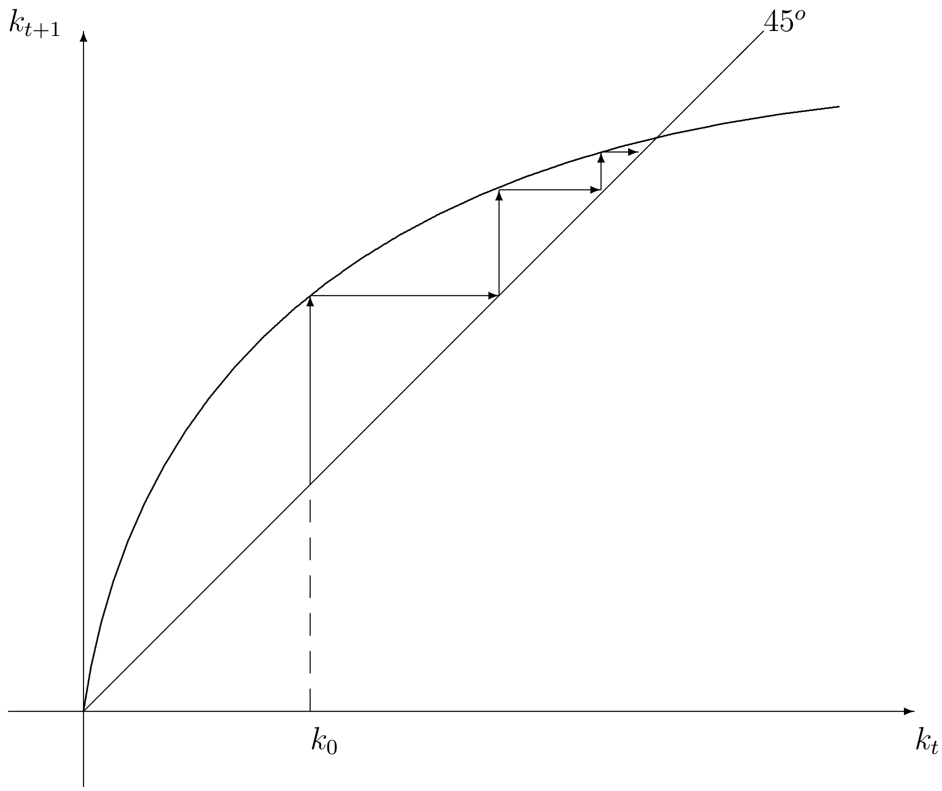
there are forces that would drive it away. For this to occur, we need the derivative of (13) have an absolute value of less than 1, $\frac{\partial k_{t+1}}{\partial k_t} \leq 1$. Looking at equation (13), due to the ambiguity of s_r, there is likewise no guarantee that any equilibrium steady state is stable, without more assumptions and restrictions. The intuition is as follows, let k^* be the steady state stock of capital. Suppose an economy begins with $k_0 < k^*$. In this state of the world, the low capital stock necessarily implies a high rental rate, and a low wage rate. If rental rates has a negative effect of savings, this would imply that savings would fall, or be low. The latter scenario implies that capital stock would continue to deviate from the steady level, without having the opportunity to reach it.

Nonetheless, if the dynamic stability conditions are fulfilled, and there exists at least one steady state commiserating with a positive capital stock, the economy will converge to the state such as in figure 2. This could be driven by increases in population growth n or increase in effectiveness g of the labour force. The condition for dynamic stability is,

$$0 < \quad \frac{-s_w k_t f''(k_t)}{(1+n)(1+g)-s_r f''(k_{t+1})} \quad < 1$$

What is particularly interesting here is that although future generations never meet, and all agents pursue their own selfish objectives, when the dynamic stability condition is met, each generation drives towards the steady state equilibrium.

Figure 2: Dynamic Adjustment



1.3 Command Optimum

We will now examine whether a social planner maximizing a social welfare function consisting of a discounted utility of individual utility would choose a allocation that differs from the decentralized equilibrium we have just discussed. Specifically, we will make the following assumptions.

1. The social planner cares for a finite sequence of individuals across time, this terminal time being T.
2. The discount rate to social planner for future generations is R. If the social planner cares less for future generations, then $R > 0$, and if she cares equally for all generations, $R = 0$. And if she weighs the utility in a "Utilitarian" fashion, for instance

by the population size, then $1+R = (1+n)^{-1}$. Notice that the effectiveness of each population cohort is not included. For the reason, you should read up on Utilitarian philosophy.

3. The general social welfare function is,

$$U = \frac{u(c_{2,0})}{1+\rho} + \sum_{t=0}^{T-1} \frac{u(c_{1,t}) + \frac{u(c_{2,t+1})}{1+\rho}}{(1+R)^{t+1}} \tag{14}$$

4. The resource constraint in each period is,

$$\begin{aligned} K_t + F(K_t, A_t L_t) &= K_{t+1} + A_t L_t c_{1,t} + A_{t-1} L_{t-1} c_{2,t} && (15) \\ \Rightarrow k_t + f(k_t) &= k_{t+1}(1+n)(1+g) + c_{1,t} + \frac{c_{2,t}}{(1+n)(1+g)} && (16) \end{aligned}$$

With the assumptions out of the way, we can now solve for the social planner's problem where she maximizes the social welfare function of equation (14) by choosing consumption levels in retirement for each cohort, and their respective investment levels, which in turn would determine the first period consumption levels. This choice necessarily is subject to the accumulation constraint of equation (16), and the fact that the initial level of per effective capita capital k_0 and terminal level of capital k_{T+1} are both given. Note that the latter can be thought of as the terminal target if the weight given to $T+1$ is positive since we have assumed that the social planner is concerned primarily with periods 0 to T. Alternatively, if she does not place any weight on T, then k_{T+1} is just zero.

In the current form, it is easier to simply substitute the constraint of equation (16) into the social welfare function so that we can write equation (14) as,

$$\begin{aligned} & \cdots + \frac{u(c_{1,t-1}) + u(c_{2,t})(1+\rho)^{-1}}{(1+R)^t} + \frac{u(c_{1,t}) + u(c_{2,t+1})(1+\rho)^{-1}}{(1+R)^{t+1}} + \cdots \\ = \; & \cdots + \frac{u(k_{t-1} + f(k_{t-1}) - k_t(1+n)(1+g) - c_{2,t-1}(1+n)^{-1}(1+g)^{-1}) + u(c_{2,t})(1+\rho)^{-1}}{(1+R)^t} \\ & + \frac{u(k_t + f(k_t) - k_{t+1}(1+n)(1+g) - c_{2,t}(1+n)^{-1}(1+g)^{-1}) + u(c_{2,t+1})(1+\rho)^{-1}}{(1+R)^{t+1}} + \cdots \end{aligned} \tag{17}$$

Therefore, to find the optimal choices of the social planner, we can find $c_{2,t}$ and k_t that maximizes the social welfare function. The consequent first order conditions are,

$$\begin{aligned}
\frac{u'(c_{2,t})}{(1+\rho)(1+R)^t} - \frac{u'(c_{1,t})}{(1+n)(1+g)(1+R)^{t+1}} &= 0 \\
\Rightarrow \frac{u'(c_{2,t})}{(1+\rho)} - \frac{u'(c_{1,t})}{(1+n)(1+g)(1+R)} &= 0 \qquad (18) \\
-\frac{u'(c_{1,t-1})(1+n)(1+g)}{(1+R)^t} + \frac{u'(c_{1,t})(1+f'(k_t))}{(1+R)^{t+1}} &= 0 \\
\Rightarrow -u'(c_{1,t-1})(1+n)(1+g) + \frac{u'(c_{1,t})(1+f'(k_t))}{1+R} &= 0 \qquad (19)
\end{aligned}$$

Naturally, equation (18) states the optimizing condition for consumption to be allocated between two generations in any period, essentially that the marginal utility of consumption to the younger cohort should equate with that of the older retired cohort. Equation (19) on the other hand states the optimizing condition for the intertemporal allocation of resources/capital between two periods for a particular cohort, in other words, the optimal savings scheme. In fact when you put the two conditions together, you obtain the optimal choice determined by the individuals independently in a competitive framework. To be precise, substitute $u'(c_{1,t})$ from equation (18) into equation (19),

$$\begin{aligned}
u'(c_{1,t-1})(1+n)(1+g)(1+R)(1+\rho) &= (1+f'(k_t))u'(c_{2,t})(1+n)(1+g)(1+R) \\
\Rightarrow u'(c_{1,t-1})(1+\rho) &= (1+f'(k_t))u'(c_{2,t}) \qquad (20)
\end{aligned}$$

which is precisely the condition of equation of (8), with $f'(k_t) = r_t$. This outcome is not surprising given that the social planner has essentially simply optimized the weighted welfare of different generations.

A good question to ask at this juncture is the implications of the optimum condition of equation (8) or (20), particularly in relation to the *Golden Rule* and *Modified Golden Rule* we had derived prior. Let the optimal steady state levels of consumption and capital per effective capita be denoted by c_1^*, c_2^* and k^* respectively. Notice that we have dropped the subscript for time, since in steady state, they are unchanged. Substituting these values into equation (18) and (19), we obtain,

$$\begin{aligned}
\frac{u'(c_2^*)}{(1+\rho)} &= \frac{u'(c_1^*)}{(1+n)(1+g)(1+R)} \qquad (21) \\
u'(c_1^*)(1+n)(1+g) &= \frac{u'(c_1^*)(1+f'(k^*))}{1+R} \\
\Rightarrow (1+n)(1+g)(1+R) &= 1+f'(k^*) \qquad (22)
\end{aligned}$$

The key equation here is equation (22) which for moderate values of R, n, and g implies that,

$$f'(k^*) \approxeq n + g + R \tag{23}$$

which is very similar to the *Modified Golden Rule* we obtained in the Infinite Horizon model of Cass (1965), we the difference being replacing the private discount rate ρ with the social discount rate R. This being unsurprising since there is no reason to believe that the social planner's discount rate would equate with the private one. There is no difference between the private choice and social planner's choice in the Infinite Horizon model of Cass (1965) because each individual is infinitely lived. Whereas in the current setting, because each individual lives for only 2 periods, there is the selfish individual does not consider his choices on future generations. Finally, if the social planner weights each generation equally, so that $R = 0$, we obtain the *Golden Rule* of Solow (1956).

1.3.1 A Turnpike Theorem

We will now examine under what conditions would the social planner's choice of capital accumulation converge towards the steady state equilibrium. The idea is that equations (16), (18), and (19) define what must happen in each and every period under the command optimum, where as noted in the previous section, equation (18) defines the optimal allocation between the young and old cohort in any period, while equation (16) defines the condition of intertemporal allocation. Since we are principally concerned here with the latter, we will use equation (16) subject to the resource constraint to examine the conditions under which the capital stock would converge to the optimal choice k^*. Since equation (16) is nonlinear, we will linearize it using Taylor series expansion around the optimal steady state value of k^*.

$$\begin{aligned}
&-u'(c_1^*) + \frac{u'(c_1^*)(1+f'(k^*))}{(1+n)(1+g)(1+R)} \\
&-u''(c_1^*)\left[(1+f'(k^*))(k_{t-1}-k^*) - (1+n)(1+g)(k_t-k^*)\right] \\
&+\frac{u''(c_1^*)(1+f'(k^*))\left[(1+f'(k^*))(k_t-k^*) - (1+n)(1+g)(k_{t+1}-k^*)\right]}{(1+n)(1+g)(1+R)} \\
&+\frac{u'(c_1^*)f''(k^*)(k_t-k^*)}{(1+n)(1+g)(1+R)} = 0
\end{aligned}$$

First note that the first line in the expansion is just the first order condition at the optimal values, and consequently will be equal to zero. For the rest of the derivation, we will denote $u''(c_i^*) = u_i''$, and $f'(k^*) = f'$. Therefore the last equation reduces to the following,

$$
\begin{aligned}
\Rightarrow \quad & -u_1''(1+f')(k_{t-1}-k^*) - \frac{u_1''(1+f')}{1+R}(k_{t+1}-k^*) \\
& + \left[u_1''(1+n)(1+g) + \frac{u_1''(1+f')^2 + u_1' f''}{(1+n)(1+g)(1+R)}\right](k_t - k^*) = 0 \\
\Rightarrow \quad & (1+R)(k_{t-1}-k^*) + (k_{t+1}-k^*) \\
& - \left\{\left[\frac{(1+n)(1+g)(1+R)}{1+f'} + (1+R)\right] + \frac{u_1' f''(1+R)}{u_1''(1+f')^2}\right\}(k_t - k^*) = 0 \\
\Rightarrow \quad & (1+R)(k_{t-1}-k^*) + (k_{t+1}-k^*) - \left[2 + R + \frac{u_1' f''(1+R)}{u_1''(1+f')^2}\right](k_t - k^*) = 0 \\
\Rightarrow \quad & (1+R)(k_{t-1}-k^*) + (k_{t+1}-k^*) - \left[2 + R + \frac{u_1' f''}{u_1''(1+f')(1+n)(1+g)}\right](k_t - k^*) = 0
\end{aligned}
$$

The first equality above follows from dividing throughout the expansion to eliminate the coefficient of $(k_{t+1} - k^*)$. The final equality follows from the use of equation (22) which is equation (19) at the steady state. Let $\alpha = \frac{u_1' f''}{u_1''(1+f')(1+n)(1+g)} > 0$, then we can write the final equality as,

$$(1+R)(k_{t-1}-k^*) - [2+R+\alpha](k_t - k^*) + (k_{t+1}-k^*) = 0 \tag{24}$$

Equation (24) is just a homogeneous second order difference equation, the solution method of which is similar to a homogeneous second order differential equation. The characteristic equation to equation (24) is,

$$x^2 - (2+R+\alpha)x + (1+R) \quad = \quad 0 \tag{25}$$

The characteristic equation has two roots, both of which are positive. One way to see this, first notice that for $x = 0$, the characteristic equation is positive. Next note that the function is convex, in other words, it is first decreasing, then increasing. Notice then that the function first intersects the horizontal axis before $x = 1$ since at $x = 1$, the characteristic equation has a value of $-\alpha$. Finally, the convexity ensures the function will have a second root, which occurs after $(1 + R)$ since at $x = 1 + R$, the value of the characteristic equation is $-\alpha(1 + R) < 0$.

Denoting the two roots as x_1 and x_2, with the former being the smaller, we can now consider the path that the capital stock will take, which is written (as in the differential equation case) as,

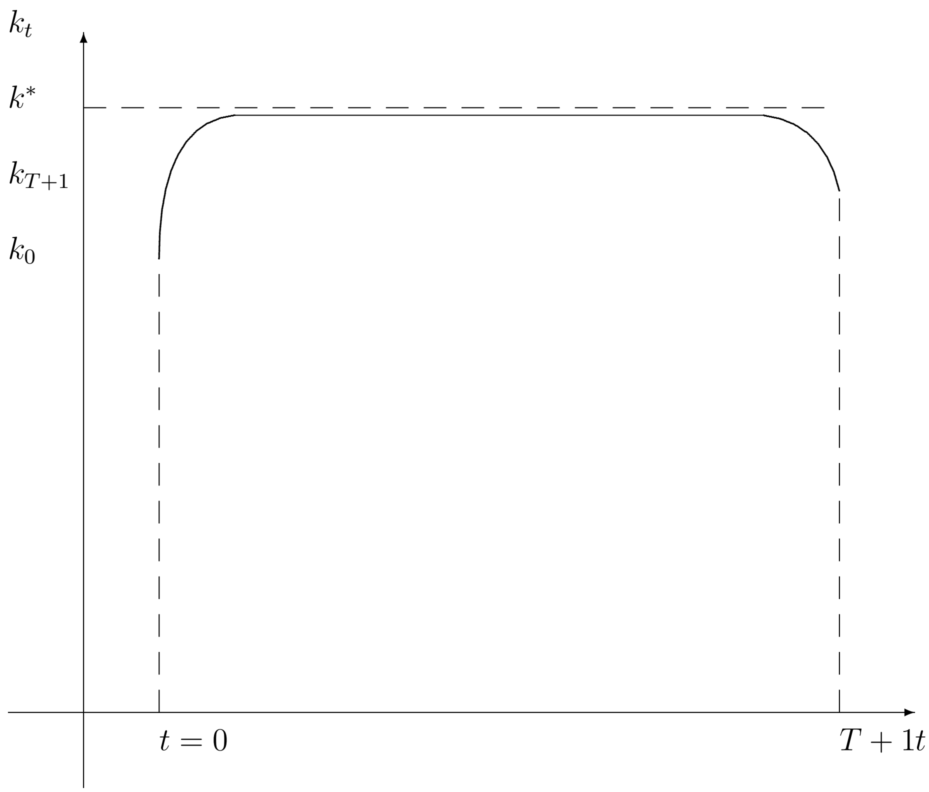
$$k_t - k^* \quad = \quad A_1 x_1^t + A_2 x_2^t \tag{26}$$

where A_1 and A_2 are the constants of the optimal path that need to be definitized through the initial and terminal conditions. Without loss of generality, let both k_0 and k_{T+1} be less than k^*, and let $k_0 < k_{T+1}$. Then at $t = 0$ and $t = T+1$ (Recall that the social planner is not concerned with $t = T+1$. Her interest ceases at $t = T$, but $t = T+1$ is some terminal condition given to her.) respectively,

$$\begin{aligned} k_0 - k^* &= A_1 + A_2 \\ k_{T+1} - k^* &= A_1 x_1^{T+1} + A_2 x_2^{T+1} \end{aligned} \tag{27}$$

Without solving for the definite values, note first that as $T \to \infty$, $x_2^{T+1} \Rightarrow \infty$ since $x_2 > 1 + R$ which means that for $k_{T+1} - k^*$ be finite, A_2 has to be close to zero. Given this fact, this in turn implies that A_1 must be close to the difference between $k_0 - k^*$. Diagrammatically, the path is represented in figure 3, What figure 3 reveals is that initial

Figure 3: Optimal Path of Capital Stock; Turnpike Theorem



at low levels of t, the effect of A_1 dominates so that the economy tends to the steady state quickly, and remains at k^* (modified Golden Rule level) until $t \to T+1$, since it has to meet the terminal condition (or target). The diagram shows what is typically known as the *Turnpike Property*. Intuitively, it says that the best way to get from k_0 to k_{T+1} is to stay close to k^*. Put another way, it says that even in a finite horizon problem, the economy should stay close to the Modified Golden Rule for as long as possible.

A relevant question you may have in your mind is whether our analysis generalizes to the infinite case, such as was discussed in our discussions of Calculus of Variations and Optimal Control Theory. The answer is dependent on the social planner's preference for the future generations. If $-1 < R < 0$, then in effect, the social planner values future generations more than the current. In such a case, it is clear that the social planner's objective (14) will not converge, in other words, there is no optimal solution. On the other extreme where she values the future generations less, such as when $R > 0$, there will be convergence of U of objective (14) as long as the rate of change of the "social discount factor" is greater than that of the individual instantaneous utility function realizations. Strictly speaking, if $R = 0$, U does not converge. However, it has been argued that we can treat treat the solution to the infinite horizon case as that of taking the limit of the finite horizon case, for $T \to \infty$. Note that as long as $R > 0$, k^* is associated with the *Modified Golden Rule*, while for $R = 0$, k^* is that associated with the *Golden Rule* similar to what we have discussed in Solow (1956).

It is important to realize here that unlike Cass's (1965) infinite horizon model we had discussed earlier, we have here that R for the social planner's problem, which is arbitrary in nature, differing from the individual. This is because in the infinite horizon, the individual is a representative constituent of the economy, with a infinitely long planning horizon. There is no reason for the social planner to have a different discount factor. On the other hand, the social planner's choice will inherently differ from that of the decentralized economy here since the planning horizons are inherently different. But that in and of itself does not imply any inefficiency, but a difference in "opinion". Nonetheless, given $R \geq 0$, which implies the fact that k^* will be below the Golden Rule level, the economy will be Pareto Efficient. Inefficiency arises only when the economy has an over-accumulation of capital so that its reduction would result in an increase in aggregate utility. For a discussion, you can read Blanchard and Fischer (2000).

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