Quantitative Methods in Economics
ECON 271:10
Chapter 1
Introduction

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1 Real Numbers

1.1 Sets

A *set* in mathematics is simply a collection of objects that we consider to form a whole (category/grouping). The constituents s of that set S are called its *elements*. A set that has no elements is known as an *empty set* or *null set*, and is denoted by $\varnothing$. If s_i is an element of S, we can denote it as $s_i \in S$, and is read as "s_i is an element of S" or "s_i is in S". In turn, if a_j is an element that is not in S, we denote that as $a_i \notin S$, which is read as "a_i is not an element of S" or "a_i is not in S".

Suppose the set A has the following elements,

$$2, 3, 5, 7$$

We can then denote this fact by the following notation,

$$A = \{a | a \text{ is a prime number less than } 10\}$$

which is read as "A is the set of all a such that a is a prime number less than 10". Alternatively, we can denote the idea by,

$$A = \{2, 3, 5, 7\}$$

and we read this as "A is the set consisting of the elements $2, 3, 5, 7$.

Let us consider the two sets below:

$$A_1 = \{2, 3\} \qquad A_2 = \{2, 3, 5\}$$

where each element of A_1 is found in A_2. When this occurs, we say that A_1 is a subset of A_2, and we denote it as $A_1 \subset A_2$ or $A_1 \subseteq A_2$. It must be noted that the empty set is also commonly considered to be part of any set, so that $\varnothing \subset A_i$, where $i = \{1, 2\}$. That is,

$$\varnothing \subset A_1$$
$$\& \quad \varnothing \subset A_2$$

Let us consider now some important sets of numbers. The set of *all integers* is $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. In turn, the set of *nonnegative integers* is $\{0, 1, 2, \ldots\}$, and the set of *negative integers* is $\{-1, -2, \ldots\}$, both of which are subsets of all integers.

Rational numbers are ratios of integers. For example, for integers a and b, $\frac{a}{b}$ is a rational number, where a is the *numerator*, and b is the *denominator*. You may ask then, are there numbers that are not rational, and the answer is "Yes". For example, $\sqrt{2}$ is not a rational number, but it occurs naturally as the length of the hypotenuse of an isoceles right angle triangle with equal length for the opposite and adjacent "legs" of 1. Another example of numbers that are not rational is π, which can be described by the ratio of a circles circumference over its diameter. These numbers are known collectively as *irrational numbers*. Both rational and irrational numbers then together in their entirety forms the *set $\mathbb{R}$ of real numbers*, and each element can be written in *decimal representation*, for example, $\frac{1}{4} = 0.25$ and $\frac{1}{3} = 0.3333\ldots$, the former being *terminating* type real numbers, while the latter is the *repeating* type real numbers. Of course, if you are sharp, these two types do not cover all rational numbers, since $\sqrt{2}$ and π are neither repeating nor terminating, and there are infinitely many of such types.[1] You may also wonder how do

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[1]There are other number systems, which we will barely touch on in this course, which if you are interested in, you may find out more about at your convenience, such as *complex numbers*.

you accurately denote real numbers that neither repeat nor terminate. In those cases, we use the symbol ≈, which is read "approximately equal to", examples of which are,

$$\sqrt{2} \approx 1.4142 \qquad \pi \approx 3.1416$$

By convention, it is sufficient to represent real numbers up to 4 decimal places.

1.2 Coordinates

Diagrammatically, real numbers can be and often is represented on a horizontal line. The real number 0 anchors this representation as the *origin*. Then once we have decided on the scale by marking off segments of equal length on either side of 0, we can associate the other integers to this origin such as in figure 1.

Figure 1: Real Line

Further by subdividing the segments, we can locate both the rational and irrational numbers such as on figure 2, so that every point on the real line $\mathbb{R}$ is associated with a **unique** real number, say x, which we call the *coordinate of* $\mathbb{R}$.

Figure 2: Real Line with Rational & Irrational Numbers
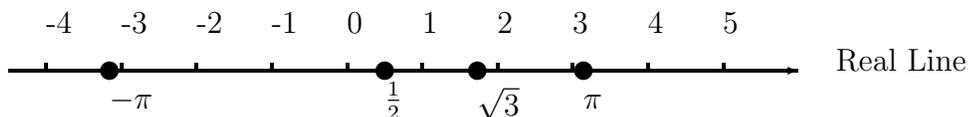
1.3 Inequalities

As you would have noticed, and perhaps wondered, how the real numbers are ordered on the figures above. Indeed, as we move on the real line from the left to right, the numbers increase in value. To describe the relationships between the real numbers, we use the symbols < (less than), > (greater than), ≤ (less than or equal to), and ≥ (greater than or equal to), which are known as *inequalities*. Formerly,

Definition 1 *If a and b are real numbers, then*

$$a < b \quad \text{means} \quad b - a > 0 \ (b - a \text{ is positive})$$

$$a \leq b \quad \text{means} \quad a < b \text{ or } a = b$$

Of course, you may read the definition with the variables switched and the signs similarly switched. For example, $a < b$ is the same as saying $b > a$, and $a \leq b$ is the same as saying $b \geq a$.

As you would have known by now from all your math education, that symbols have properties that in their using, they must be adhered to. Similarly, inequalities have the following properties:

1. **Addition Property**: If $a \leq b$, then $a + c \leq b + c$, for any choice of c.

2. **Multiplication Property**:

    (a) If $a \leq b$ & $c > 0$, $\Rightarrow$ (then) $a.c \leq b.c$

    (b) If $a \leq b$ & $c < 0$, $\Rightarrow$ (then) $a.c \geq b.c$

3. **Division Property**:

    (a) If $a > 0$, $\Rightarrow \frac{1}{a} > 0$

    (b) If a & b are both positive or both negative, and $a < b$, $\Rightarrow \frac{1}{a} > \frac{1}{b}$

4. Trichotomy Property: For any 2 real numbers a & b, one and only one of the following is true: $a < b$, $a = b$, $b > a$

5. Transitive Property: If $a < b$ & $b < c$, $\Rightarrow a < c$

Finally, note that these symbols and properties can be combined. For example, if $a < b$ and $b < c$, we can write this comprehensively as $a < b < c$. Geometrically, the last inequality says that b is to the right of a, and c is to the right of both a and b.

1.4   Variables

In Mathematics (and any field of study that uses Math, including Economics), a v*variable* is a symbol (usually Latin alphabets, x, y, ..., or Greek alphabets, α, β, ...) that is used to denote a real number (that perhaps is unknown in value to us for the moment). An

inequality, as we have noted, is a mathematical statement involving one or more variables, and one of the previously discussed inequality symbols ($<, >, \leq, \geq$). When we wish to *solve* an inequality for the unknown variable(s), we **find all possible numbers that the variable can take so as to ensure the inequality statement is always true**. This set of numbers is thus known as the *solution* or *solution set*. If two or more inequalities give the same solution, we call them *equivalents*.

How then do we begin to solve for the solution given an inequality? We can do so by applying the properties discussed prior, and sequentially and systematically obtain a sequence of equivalent inequalties that (hopefully) lead us to the solution.

Example 1 *Solve the inequality $x + 1 \leq 2x - 3$.*

Solution 1

$$x + 1 \leq 2x - 3$$
$$\Rightarrow \quad 1 \leq x - 3 \qquad \text{subtract } x \text{ from both sides}$$
$$\Rightarrow \quad 4 \leq x \qquad \text{add 3 to both sides}$$
$$\Rightarrow \quad x \geq 4$$

Therefore the solution is the set of all real numbers to the right of 4, inclusive of 4. Note that all the 4 inequalities are equivalents.

Of course not all inequalities are that easy to solve. Consider the following example.

Example 2 *Solve $x^2 + 2x - 15 > 0$*

Solution 2 *Factoring the inequality, we obtain,*

$$(x - 3)(x + 5) \quad > \quad 0$$

You can think of $(x - 3)$ and $(x + 5)$ as real numbers should you know the solution. Thus, we know the inequality holds true if $(x - 3) < 0$ and $(x + 5) < 0$, and if $(x - 3) > 0$ and $(x + 5) > 0$. When $(x - 3) < 0$ and $(x + 5) < 0$ is true, we have $x < 3$ and $x < -5$, so that $x < -5$ must be true. Further, for $(x - 3) > 0$ and $(x + 5) > 0$ to be true, we need $x > 3$ and $x > -5$, so that $x > 3$. This thus mean that the solution set includes all values of x such that either $x < -5$ or $x > 3$. This can be written succinctly as $\{x | x < -5 \text{ or } x > 3\}$.

For completeness, consider how this method of solution could be used to solve inqualities involving ratios.

Example 3 *Solve $\frac{x-2}{x+7} > 3$.*

Solution 3

$$\frac{x-2}{x+7} > 3$$

$$\Rightarrow \quad \frac{x-2}{x+7} - 3 > 0 \qquad \text{subtract 3 from both sides of inequality}$$

$$\Rightarrow \quad \frac{(x-2)-3(x+7)}{x+7} > 0 \quad \text{express with common denominator}$$

$$\Rightarrow \quad \frac{x-2-3x-21}{x+7} > 0$$

$$\Rightarrow \quad \frac{-2x-23}{x+7} > 0 \qquad \text{Simplify expression}$$

Then the inequality is true if either $-2x - 23 > 0$ and $x + 7 > 0$ or $-2x - 23 < 0$ and $x + 7 < 0$. Looking at each in turn. Notice that for the first two inequalities, we then obtain $x < -\frac{23}{2} = -11\frac{1}{2}$ and $x > -7$ which cannot both be true. So that this case is not possible. That leaves the last two inequalities, which yields $x > -\frac{23}{2}$ and $x < -7$, or more succinctly, $-\frac{23}{2} < x < -7$, which is the solution set.

1.5 Absolute Value

The *absolute value* of a number, is the distance of that number from 0 on the real line. Thus the absolute value of -5 and 5 is 5. Formally,

Definition 2 *The absolute value of a real number x, which is denoted commonly by $|x|$, is defined as,*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This thus means that the absolute value of any number is either positive or 0. Some properties of absolute value include the following:

   If a and x are real numbers, then:

1. $|a - x| = |x - a|$

2. $|ax| = |a|.|x|$

3. $-|x| \leq x \leq |x|$

4. $\left|\frac{a}{x}\right| = \frac{|a|}{|x|}$ for $a \neq 0$

5. $|x|^n = |x^n|$ for an integer n.

Another important property is that **if the absolute value of an expression equals some positive number, say α, then the expression itself is equal to either α or** $-\alpha$. Therefore, $|x| = 7$ means that $x = 7$ or $x = -7$.

Note then the equivalency of the following:

1. $|x| < a$ is equivalent to $-a < x < a$

2. $|x| \leq a$ is equivalent to $-a \leq x \leq a$

3. $|x| > a$ is equivalent to $x < -a$ and $x > a$

4. $|x| \geq a$ is equivalent to $x \leq -a$ and $x \geq a$

Consider how these ideas can be used in mathematics and statistics:

Example 4 *Find all values of x for which $|x - 2| \leq 3$.*

Solution 4 *By definition,*

$$|x - 2| \leq 3$$
$$\Rightarrow -3 \quad \leq x - 2 \leq \quad 3 \text{ by adding 2 to the inequality}$$
$$\Rightarrow -1 \quad \leq x \leq \quad 5$$

So the solution is $\{x| - 1 \leq x \leq 5\}$.

1.6   Intervals

For two real numbers a and b, with $a < b$. A *closed interval* bounded by these two real numbers is denoted by $[a, b]$, and describes the set of all real numbers, call them x, from a to b, inclusive. In other words,

$$[a, b] \quad = \quad \{x|a \leq x \leq b\}$$

In turn, an *open interval* (a, b) consists of all real numbers x from a to b, excluding both a and b. That is,

$$(a, b) \quad = \quad \{x|a < x < b\}$$

These concepts can be combined to give combinations of intervals commonly termed *half open (semi-open)* or *half closed (semi-closed)* intervals/sets which are defined by:

$$[a, b) = \{x | a \leq x < b\} \qquad (a, b] = \{x | a < x \leq b\}$$

where in the former, a is the *left endpoint*, and in the latter, b is the *right endpoint* of each interval. Graphically, ● is used to denote an endpoint that is included in an interval, while ○ is used to denote an exluded endpoint. That is for an interval $(-1, 4] = \{x | -1 < x \leq 4\}$

Figure 3: Intervals
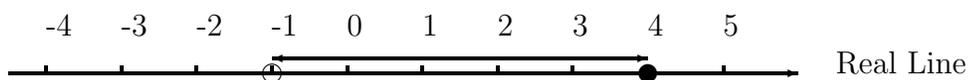
We can conceive of the existence of very larger or very small real numbers, commonly termed positive or negative infinity, denoted $+\infty$ (or simply ∞) and $-\infty$ respectively. It must be noted that they do not represent an actual real number, but rather an abstract concept we can conceive of. These can belikewise used in conjunction with actual numbers to create additional sets of real numbers. What you need to pay attention to is that since ∞, both negative and positive, are abstract concepts, so that when they are used to create intervals, they cannot represent actual endpoints. So we can define the following to complete our discussion here.

$$
\begin{aligned}
[a, +\infty) &= \{x | x \geq a\} \\
(a, +\infty) &= \{x | x > a\} \\
(-\infty, a] &= \{x | x \leq a\} \\
(-\infty, a) &= \{x | x < a\} \\
(-\infty, +\infty) &= \{x | \infty < x < +\infty\} = \mathbb{R}
\end{aligned}
$$

The prior discussion on absolute numbers and intervals now lead to an important inequality in Mathematics.

Theorem 1 ***Triangle Inequality****: For real numbers x and y,*

$$|x + y| \quad \leq \quad |x| + |y| \tag{1}$$

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which is the algebraic representation of the fact that the length of the sides in a triangle, cannot exceed the sum of the lengths of any of the two sides. Using the properties we have learned, the result can be proven.

Proof. We know that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ based on the definition and property of absolute numbers. Adding these two inequalities give us,

$$-|x| + (-|y|) = -(|x| + |y|) \; \leq (x + y) \leq \; (|x| + |y|)$$
$$\Rightarrow \quad |x + y| \leq \quad |x| + |y|$$

∎

You can show yourself another similar inequality, using likewise, the properties we have discussed.

Theorem 2 *For real numbers x and y,*

$$|x - y| \; \geq \; |x| - |y| \tag{2}$$

Proof.

$$|x| = |(x - y) + y| \; \leq \; |x - y| + |y| \qquad \text{By the Triangle Inequality}$$
$$|x| - |y| \; \leq \; |x - y|$$

∎

2 Graphing

What you will discover as we progress on, is that it is often useful to depict the equations we work with to better understand where our solution lie, or how we may go about obtaining a solution.

2.1 Rectangular Coordinates

As you may recall from your earlier education, we can depict lines (or curves) within a box bounded by a vertical and horizontal line on the left and bottom respectively, with their intersection marking the origin (such as what we did with the real line). These lines could be extended to the not only the positive, but the negative segment as well. We generally call the vertical line, the $y-axis$, and the horizontal line, the $x-axis$. And the coordinate system thus allows us to mark out any point on the *plane*. In other words, any point on this plane, call it P formed by the $x-$ and $y-$ axes, can be located by using *ordered pair* of real numbers such as in figure 4 below.

Observe that each point is reported as a couplet, that is two numbers on the real line. The combination of the signs for each number then determines which quadrant the point lies in.

With the information given by an equation, we can then obtain the coordinates to depict the lines or curves it represents on the coordinate system. This coordinate system is known as the *rectangular* or *cartesian coordinate system*. The first number, which represents a point on the $x-$axis is known as the *abscissa* of the point, and the second number represents the $y-$axis location of the point, and is known as the *ordinate* of the point.

A useful tool or idea in graphing, is to note when a line or curve intersects the axes. Where the curve intersects the $x-$axis, we call that point the $x-intercept$, and the ordinate of the point is always 0. Likewise, when a line or curve intersects the $y-$axis, we call that point the $y-intercept$, and the abscissa of those points are always 0. A line or curve can have several or no abscissa or ordinate.

Another useful concept or tool in graphing is the idea of symmetry. Symmetry can be with respect to the $x-$axis, &/or the $y-$axis, or the origin. Of course the idea of symmetry need not be exclusive to these three points, since we can always shift the idea of axes, and origin. The diagrams below provide you with a visual idea of symmetry.

Figure 4: Rectangular Coordinate System
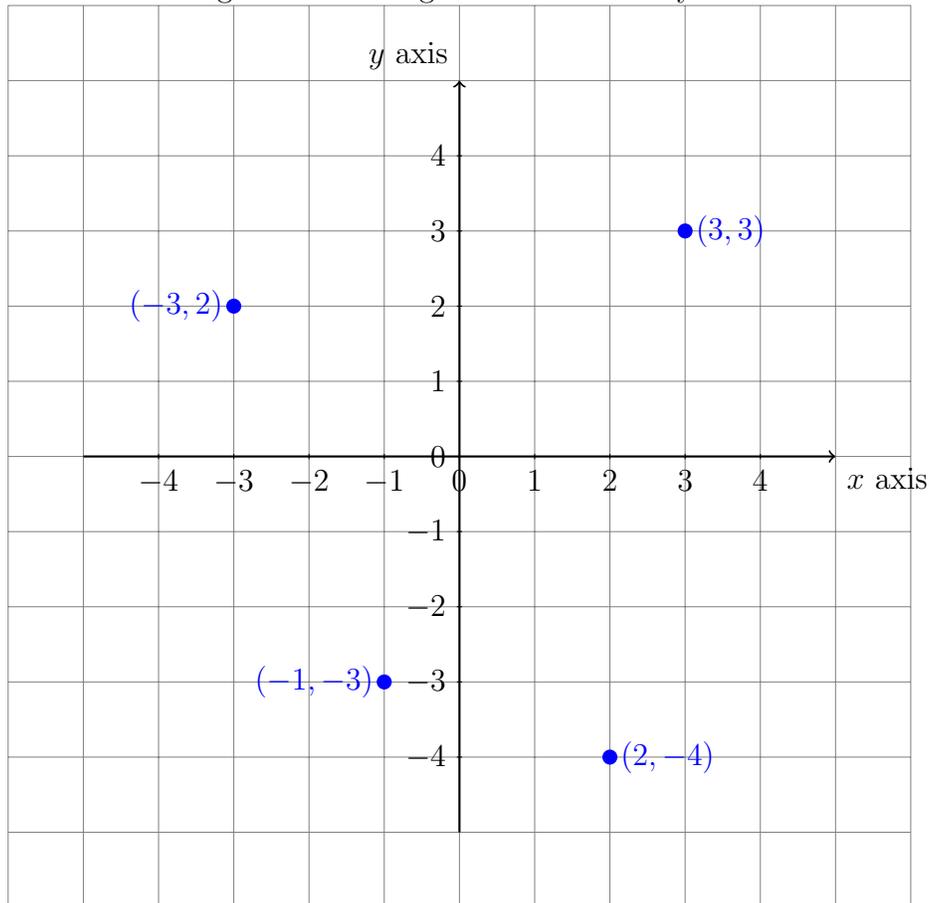
Formally,

1. *Symmetry with respect to the $x-$axis:* For every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.

2. *Symmetry with respect to the $y-$axis:* For every point (x, y) on the graph, the point $(-x, y)$ is also on the graph.

3. *Symmetry with respect to the origin:* For every point (x, y) on the graph, the point $(-x, -y)$ is also on the graph.

How can we use these latter ideas of symmetry then?

Example 5 *Consider the equation $xy^2 - x^4 + 5 = 0$ for symmetry with respect to the $x-$axis and $y-$axis.*

Figure 5: Quadrants

Quadrant 2 $x < 0, y > 0$	Quadrant 1 $x > 0, y > 0$
Quadrant 3 $x < 0, y < 0$	Quadrant 4 $x > 0, y < 0$

y axis
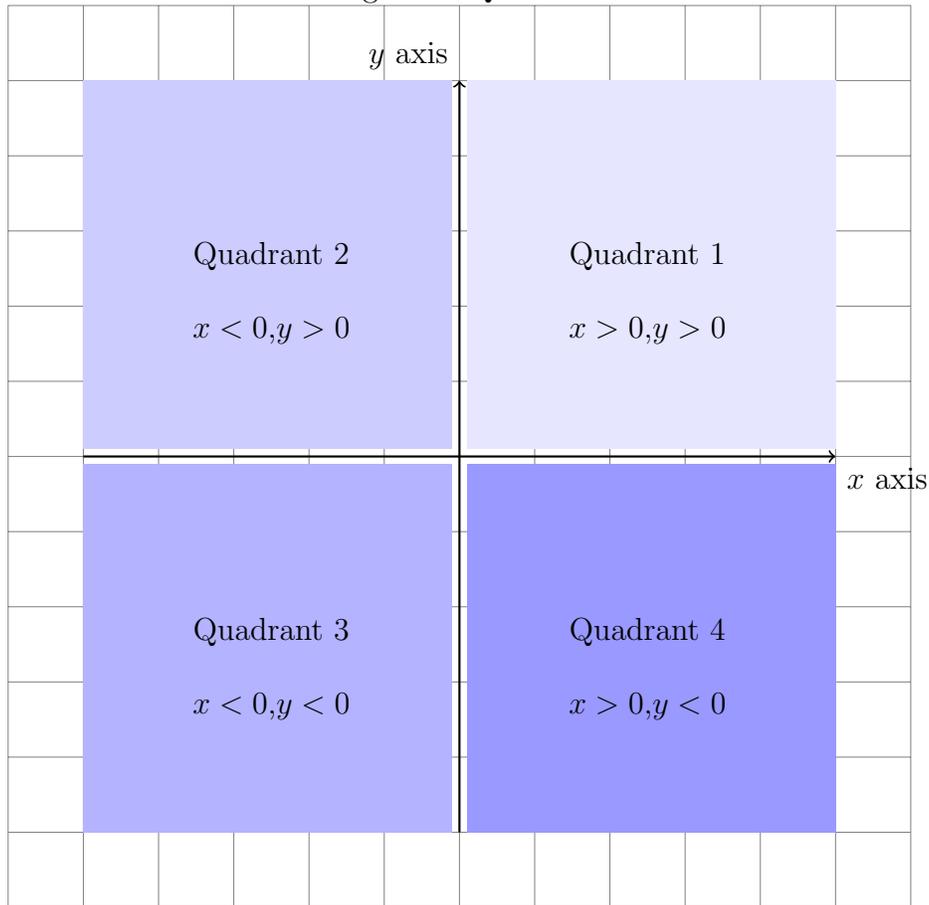
x axis

Solution 5 *For the equation to be symmetric about $x-$axis, if (x, y) is on the graph, then so too must $(x, -y)$. Then observe that both $xy^2 - x^4 + 5 = 0$ and $x(-y)^2 - x^4 + 5 = 0$ is true. Thus the graph of the equation is symmetric about the $x-$axis. It can be noted that if $xy^2 - x^4 + 5 = 0$ then $(-x)y^2 - (-x)^4 + 5 \neq 0$, so that the equation is not symmetric about $y-$axis. By the same token, neither is the equation symmetric about the origin.*

As noted above, these ideas of symmetry are not exclusive, since by simple translations, the graph can be moved to new axes, or origin. Consider $y = x^2 + 1$. This equation is similar to figure 6. All it involves is the shift of the entire graph northwards/upwards along the $y-$axis.

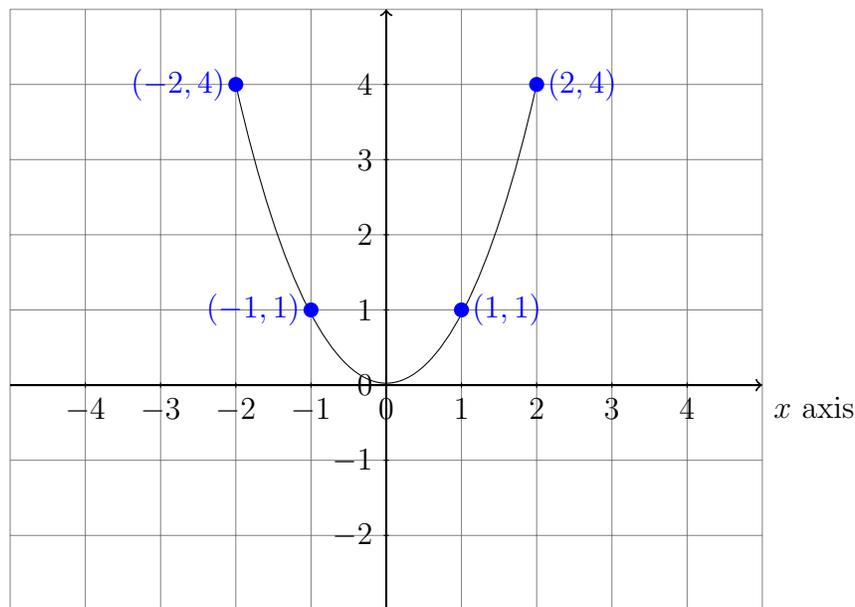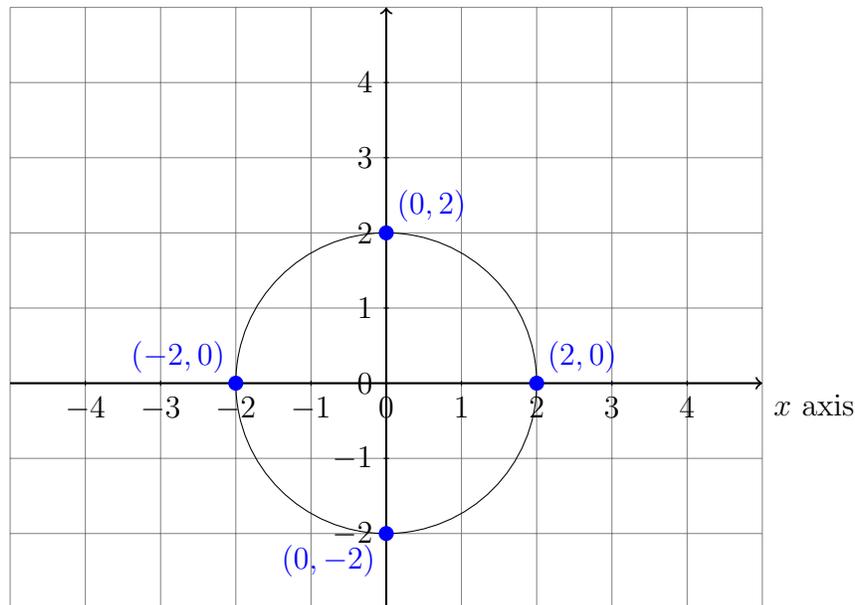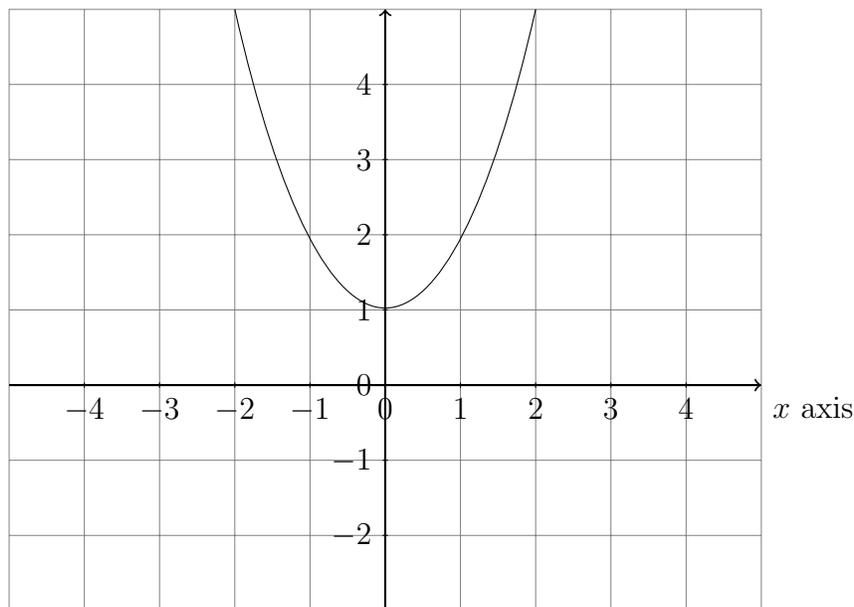Figure 6: Symmetry with respect to the $y-$axis of $y = x^2$

Figure 7: Symmetry with respect to the origin of a circle, $x^2 + y^2 = 2$

2.2 Distance Between Points

Much of what we do using mathematics, require us to quantify what we are trying to say. In other words, to give a magnitude to a statement. This requires us to measure

Figure 8: Symmetry of $y = x^2 + 1$

distances between points of interests. Let (x_1, y_1) be the coordinates of a point P_1, and let (x_2, y_2) be the coordinates of point P_2. When we move from one point to the other, say P_1 to P_2, the abscissa changes from x_1 to x_2. We denote this change in calculus as Δx. Similarly, the ordinate changes from y_1 to y_2, and that change is denoted as Δy. If the scale for both the $x-$axis and $y-$axis are the same, then the distance on the plane can be measured on the same scale, which is achieved using the *Pythagoras Theorem*.

Theorem 3 *The distance between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, denoted $|P_1 P_2|$, is*

$$|P_1 P_2| \;=\; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \;=\; \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{3}$$

The idea can be seen in figure 9 below.

It should be noted that the distance is zero if and only if P_1 and P_2 occupy the same point. Finally, since distance is never negative, $|P_1 P_2| = |P_2 P_1|$.

2.3   Circles

We can furthur use the Pythagoras Theorem to discover the *Standard Equation of a Circle* with a radius R, and centered on (h, k).
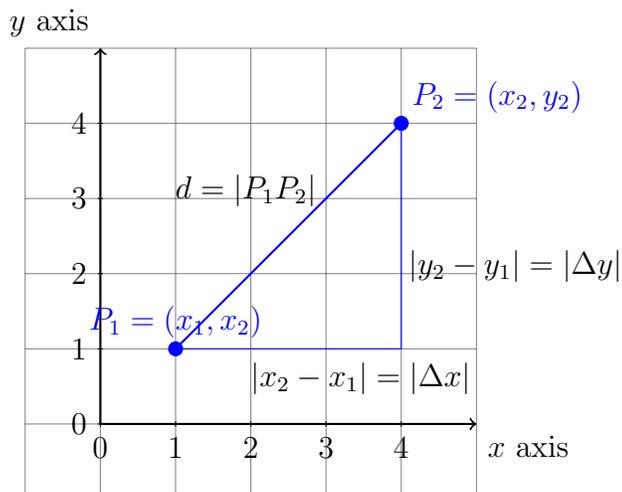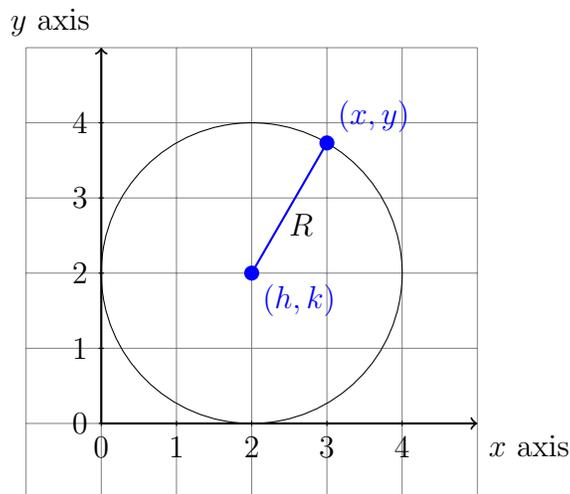
Figure 9: Pythagoras Theorem

y axis

$P_2 = (x_2, y_2)$

$d = |P_1 P_2|$

$|y_2 - y_1| = |\Delta y|$

$P_1 = (x_1, x_2)$

$|x_2 - x_1| = |\Delta x|$

x axis

Figure 10: Pythagoras Theorem

y axis

(x, y)

R

(h, k)

x axis

Given the Pythagoras Theorem we have discussed above in determining the distance between two points, we know we can write for figure 10,

$$\sqrt{(x - h)^2 + (y - k)^2} = R$$

where based on the diagram, $(h, k) = (2, 2)$. Using this idea, then

$$(x - 2)^2 + (y - 2)^2 = 4$$

is a circle centered on $(2, 2)$, with radius of 2. If instead, we were to shift the circle back

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to being centered on the origin, $(h, k) = (0, 0)$, the equation becomes,

$$x^2 + y^2 \;\; = \;\; 4$$

Indeed, the common way for expressing a circle about (a, b) is,

$$(x - a)^2 + (y - b)^2 \;\; = \;\; R^2$$

so that the values (a, b) tells you where the circle centers around. For example, if (a, b) are both negative numbers, the center is located on the third quadrant, while is $a > 0$, and $y < 0$, it would be centered in the fourth quadrant.

Problem 1 *Show that the midpoint between two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.*

3 Straight Lines

We will now give more definition for some of the different types of lines we will use in the rest of our exposition, and for mathematics in general. Simple ones first.

3.1 Gradient/Slope of a Line

Definition 3 *Let P_1 and P_2 be two distinct points with coordinates (x_1, y_1) and (x_2, y_2) respectively. Denote the line joining this two points by L. Then the slope/gradient of this line is,*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \quad \text{if} \quad x_1 \neq x_2 \tag{4}$$

If $x_1 = x_2$, the slope/gradient m is undefined, and L is a vertical line.

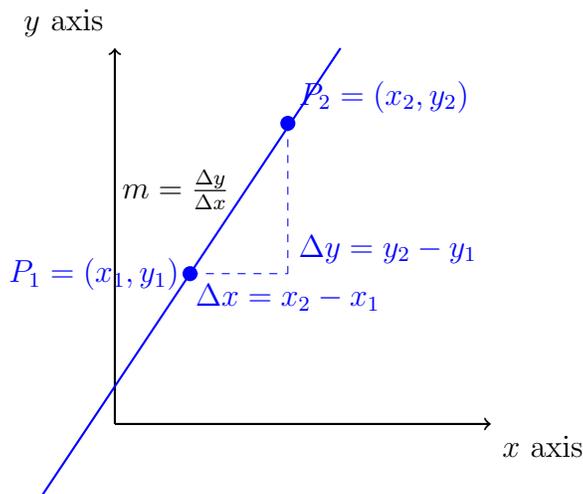This idea utilises the Pythagoras Theorem as may be observed in figure 11 below.

Problem 2 *What is the gradient of a horizontal line?*

3.2 Equations of Lines

The equation for a vertical line is

$$x \;\; = \;\; a \tag{5}$$

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Figure 11: Pythagoras Theorem

where a is a real number. How should we then describe a line with a gradient? Given our previous discussion of the gradient, then for a line with gradient m that passes through (x_1, y_1) specifically, with another point (x, y) likewise on this line, we know that,

$$
\begin{aligned}
m &= \frac{y - y_1}{x - x_1} \\
y - y_1 &= m(x - x_1)
\end{aligned}
\tag{6}
$$

where equation (6) is known as the equation of a nonvertical linie of gradient m that passes through the point (x_1, y_1) in *Point-Slope Form*.

However, some of you would be more familiar with the formulation utilising the gradient and the $y-$intercept. In other words, (x_1, y_1) is replaced by the intercept, which may be denoted by $(x_1, y_1) = (0, b)$, where b is a real number. Replacing $(0, b)$ for (x_1, y_1) gives us the equation of a straight line in *Slope-Intercept Form*.

$$
y = mx + b
\tag{7}
$$

3.3 Intersecting Lines; Parallel & Perpendicular Lines

For two distinct lines L_1 and L_2, if there is one point P in common between them, then we say that the two lines *intersect* and P is their *point of intersection*. As you should know by now, this is quite easy to find, given your knowledge of basic algebra.

Example 6 *Find the point of intersection between $x + y = 5$ and $2x + y = 6$.*

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Solution 6 *Since we know that at the point of intersection, the abscissa must be the same. That is $x = 5 - y$ from the first equation must give you the same real number as $x = \frac{6-y}{2}$ from the second. Putting this down, we have*

$$
\begin{aligned}
5 - y &= \frac{6 - y}{2} \\
\Rightarrow 10 - 2y &= 6 - y \text{ multiplying 2 to both sides of the equation} \\
\Rightarrow y &= 4
\end{aligned}
$$

which gives us the value of y at the intersection. In turn, when $y = 4$, from either equation, we get $x = 1$. Therefore the coordinates of the point of intersection is $(1, 4)$.

The technique used above to obtain the solution is known as the *substitution technique.*

You might now ask if there are incidences when two distinct lines have no intersection. And, there is, and such lines, as you must now know are known as *parallel lines.*

Theorem 4 *Two distinct parallel lines are parallel if and only if they have the same gradient.*

Proof. To proof such a statement of theorem with the qualifier **if and only if**, we need to prove both ends of the statement.

To proof the "if" portion, let $y = mx + b$ and $y = mx + c$ be equations of two lines with the same gradient m, and where their intercept $b \neq c$. These two equations have no common point and in consequence do not intersect. Therefore two lines with the same gradient are parallel.

To proof the "only if" portion, consider two lines with the following equations, $y = mx + b$ and $y = nx + c$, where $m \neq n$ (and b and c are possibly the same). We can then obtain their intersection point, which is at $x = \frac{c-b}{m-n}$ and $y = \frac{mc-nb}{m-n}$, and they are not parallel. ■

The gradient can be used to determine if two lines are perpendicular. In other words, the two lines meet each other at right angles.

Theorem 5 *Two nonvertical lines are perpendicular if and only if the product of their gradient is -1. This thus mean that for two lines with gradient m and n, they are perpendicular if and only if,*

$$
\begin{aligned}
m.n &= -1 \\
\Rightarrow m &= -\frac{1}{n}
\end{aligned}
$$