Quantitative Methods in Economics
ECON 271:10
Chapter 2
Functions & Limits

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1 Functions

In mathematics, statistics, and economics, and their applications, you would observe a correspondence between two sets of numbers. For example, the volume V of a sphere of radius r, which is given by the formula (correspondence) $V = \frac{4}{3}\pi r^3$.

As a more concrete and relevant example, consider your fixed deposit [Guaranteed Investment Certificate (GIC)] at a Canadian bank, call that deposit D, paying a nominal interest of r per annum. The total value of that deposit when you withdraw it from the bank can be expressed by the formula, $y = D(1 + r)^x$, where x is the number of years/period your deposit remain in the bank, and Y is the final amount available to you when you withdraw it at the end of the agreed period. Observe that the value Y depends on x, how long you wish to deposit your monies at the bank, for a given interest rate, say 2%, and the value of your deposit D, say \$10,000. So $y = 10,000(1+0.02)^x$. Observe that if you deposit \$10,000 for $x = 1$ year, your final withdrawal will be \$10,200. If you instead withdrew in the 10^{th} year, you would get approximately \$12,190. If 10 years is too long, then a 5 year period would yield you approximately \$11,041. The key point is that if X is the set of deposit periods, and Y is the set of final withdrawal amounts, there is a one-to-one correspondence in elements of X and Y. This correspondence $y = 10,000(1 + 0.02)^x$ is known as a *function from X into Y*. More generally,

Definition 1 ***Function; Domain; Range.*** *Let X and Y be two sets of real numbers.A* ***function*** *from X to Y is a correspondence that associates with each element of X a unique element of Y. The set X is known as the* ***domain*** *of the function. For each element x in X (written as $x \in X$), the corresponding element y in Y ($y \in Y$) is known as the* ***value of the function at x, or the image of x****. The set of all images of the elements of the domain is thus known as the* ***range*** *of the function.*

Functions are often denoted by alphabets, f, F, g, G, $\ldots$, and there is in fact no strict rules as to what we may use to denote them, as long as you prespecify your intention. If f is a function from X into Y, then for $x \in X$, the corresponding image in the set Y is denoted by $f(x)$, and is read "f of x", and is the value of f at the number x. It is common to also write the value $f(x)$ in terms of its final value in Y, $y \in Y$, so that the element from the domain is coupled with its image, (x, y). In terms of the previous example, $(1, 10, 200)$. So that it is common for you to see $y = f(x)$, highlighting the correspondence between the element x in the domain with y in the range. If however, *we have some or all of the x corresponding with more than one value in the range Y, then the underlying expression is not a function.* It is also common for us to refer to x as an *independent variable*, and y as the *dependent variable*.

Functions can have also several independent variables, for example, in our compound interest example above, the interest rate r could be considered as another variable, and so too the total initial deposit D. In the case of say, $y = f(x_1, x_2, x_3)$, we can thus read it as "f of x_1, x_2 and x_3", where each of the independent variables have their own domain, $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$. And the function f maps the three variables onto $y \in Y$.

1.1 Composite Functions

Functions themselves could be build from other basic functions to obtain *Composite Functions*. Consider the function $y = (1 + x^3)^{1/3}$. It can be observed that the function is built up by raising $(1 + x^3)$ to the $\frac{1}{3}$ power. So we can write it as a composition of $y = v^{1/3}$, where $v = (1 + x^3)$.

Definition 2 ***Composition of functions.*** *Let f and g be functions. Further, suppose that x is such that $g(x)$ is in the domain of the function f. Then the function that assigns to x the value of $f(g(x))$ is known as the* ***composition of f and g****. It is denoted as $f \circ g$.*

To provide greater concreteness as to how to use these notations. If $g(x) = u$ and $f(u) = y$, then $(f \circ g)(x) = y$. We read $f \circ g$ as "f *composed with* g". Practically, the definition says to apply g first followed by f. Further, just as in the case of functions having potentially several independent variables. Composite functions can be composed by more than two functions, for example $y = (g_1 \circ g_2 \circ g_3)(x)$. It must be kept in mind that $f \circ g$ is usually not equal to $g \circ f$, so never take for granted that the order of the functions in a composition of functions can be reordered with impunity!

Definition 3 ***Even function.*** *A function f such that $f(-x) = f(x)$ is called an* ***even function****.*

In turn,

Definition 4 ***Odd function.*** *A function f such that $f(-x) = -f(x)$ is called an* ***odd function***

1.2 One-to-One Functions & Their Inverse Functions

In Economics, we typically work with functions where a distinct input of the independent variable, yields a distinct output of the dependent variable, such as in the compound interest case. That is we work with *one-to-one functions*. Precisely,

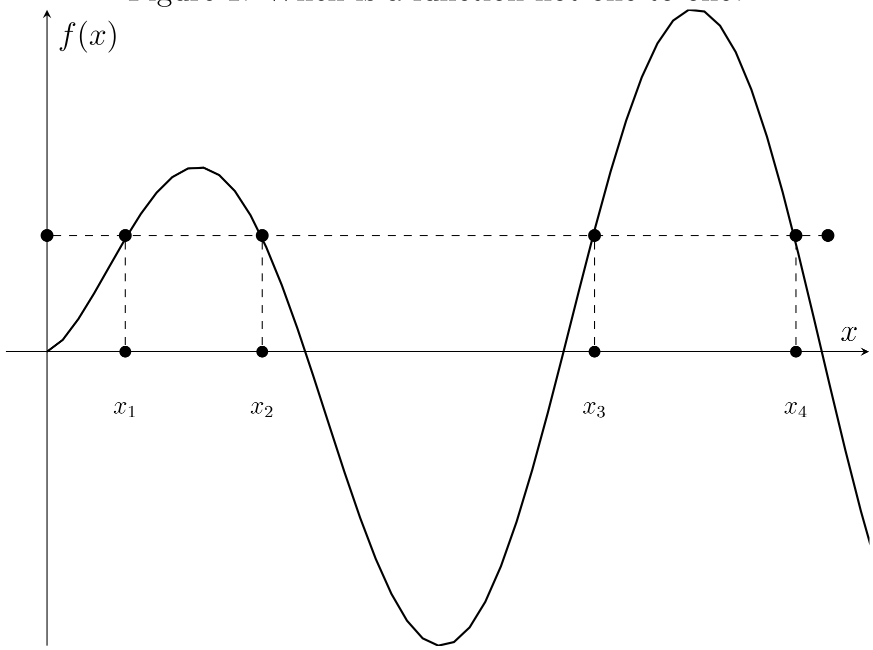
Definition 5 *A function f that assigns distinct realizations of dependent variables to distinct independent variables is known as a* ***one-to-one function****.*

So for example, $y = x^3$ is a one-to-one function for x on the entire real line, while $y = x^2$ is not. However, if we restrict $x \in [0, \infty)$, then the latter is also a one-to-one function. How do you see this graphically?

The graph of a one-to-one function has the property that *every horizontal line within the range of the function meets the graph in at most one point*. In 1, the function is not a one-to-one function since $f(x_1) = f(x_2) = f(x_3) = f(x_4)$.

The most important one-to-one function are the following, and we use these very often in economics.

Figure 1: When is a function not one-to-one?



Definition 6 *If $f(x_1) < f(x_2)$ for $x_1 < x_2$, then f is an increasing function. If $f(x_1) > f(x_2)$ for $x_1 < x_2$, then f is a decreasing function.*

These functions are broadly known as *monotonic functions*. Even if a function may not be one-to-one, we can always restrict the domain of the independent variable (the input of the function) to segments of the function which is *monotonic*.

Associated with the one-to-one function f is a mirror function g that tells us the unique independent variable that yields the output in the function f. Formally

Definition 7 *Let $y = f(x)$ be a one-to-one function. The function g that gives each output of f the corresponding unique input/independent variable is known as the* ***inverse*** *of f. In other words, if $y = f(x)$, then $x = g(y)$.*

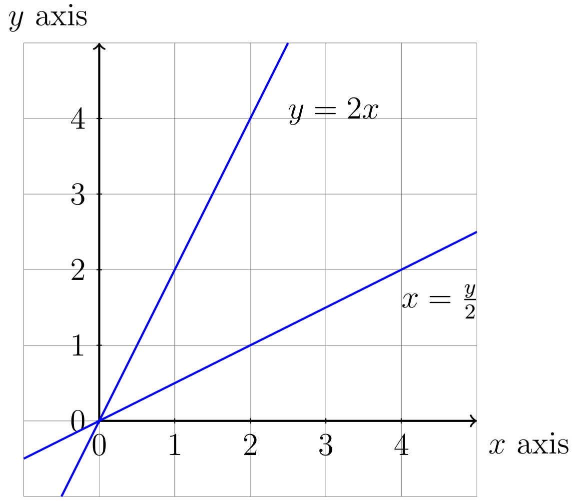
In other words, the inverse has the dependent variable becoming the input/independent variable of the function g. Often, you will see this idea reflected as follows, for a one-to-one function f.

$$y = f(x) \quad \Rightarrow x = f^{-1}(y) \quad \text{for one-to-one function } f$$

Example 1 *Find the inverse of $y = 2x$, and graph both of these functions.*

Solution 1 *It is clear that $y = 2x$ is an increasing function with gradient 2, and intercept at the origin, and is thus a one-to-one function. The inverse is thus just $x = \frac{y}{2}$ obtained by dividing 2 on both sides of $y = 2x$.*

Figure 2: Graph of $y = 2x$ & its Inverse

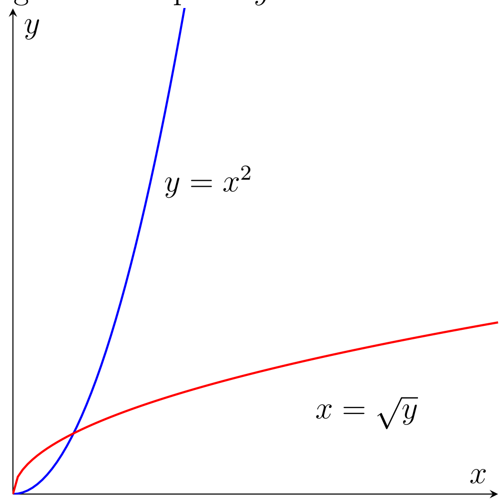


Consider another example to consolidate your understanding, now with restriction on the domain of x and in consequence y.

Example 2 *Graph $y = x^2$ and its inverse, with the domain of $x \in [0, \infty)$.*

Solution 2 *Observe that since we are restricting the domain of x to the non-negative real line, $y = x^2$ and its inverse $x = \sqrt{y}$ are both one-to-one functions.*

Figure 3: Graph of $y = x^2$ & its Inverse



2 Limits & Continuous Functions

Before we can get into derivative and integral calculus, we need to grasp the concept of *limits*. However, it should be noted, in its original form, Calculus dealt with discrete numbers, as opposed to what we have been dealing with here, with considerations of continuous real line.

2.1 The Limit of a Function

First consider the following example.

Example 3 *Let $f(x) = 2x^2$. How does $f(x)$ behave as x becomes closer and closer to 3.*

Solution 3 *Tabling the results for a subset of the realizations around 3, we get*

x	3.1	3.01	3.001	2.999	2.99	2.9
$f(x)$	19.22	18.1202	18.012002	17.988002	17.8802	16.82

Observe that the closer x gets to 3, the closer $f(x)$ gets to 18. We can thus say that "the limit of $2x^2$ as x approaches 3 is 18". This statement can be written succinctly as $\lim_{x\to 3} 2x^2 = 18$.

Now consider a more sophisticated example.

Example 4 *Let $f(x) = \frac{x^3-1}{x^2-1}$, which is not defined when $x = 1$ (why?). How does $f(x)$ behave as x gets very close to 1?*

Solution 4 *If we describe a table of values of x close to 1, such as we have done before, we could possibly have,*

x	*1.1*	*1.01*	*0.99*	*0.9*
$f(x)$	*1.5762*	*1.5075*	*1.4925*	*1.4263*

So it seems to get close to $1.5 = \frac{3}{2}$*, and yet at* $x = 1$*, it is undefined. How does that work? There are in effect two influences acting on* $f(x)$*, one on the numerator, and the other on the denominator. On both numerator and denominator, as* $x \to 1$*, both elements are pushed towards 0. And yet, as the denominator tends towards 0, we know that any number divided by a very small number becomes very large. How do these different movements balance each other out? Consider that the numerator* $x^3 - 1 = (x^2 + x + 1)(x - 1)$*, while* $x^2 - 1 = (x + 1)(x - 1)$*, so that* $f(x)$ *can be written as,*

$$\frac{x^3 - 1}{x^2 - 1} = \frac{x^2 + x + 1}{x + 1}$$

which excludes the possibility of $x = 1$ *(since we have* $(x - 1)$ *canceling out on both the numerator and denominator), so that as* $x \to 1$ *but never equaling 1 can be expressed as* $\frac{x^2+x+1}{x+1}$*. In other words,*

$$\begin{aligned} \lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} &= \lim_{x \to 1} \frac{x^2 + x + 1}{x + 1} \\ &= \frac{3}{2} \end{aligned}$$

which is what we suspected before using the table.

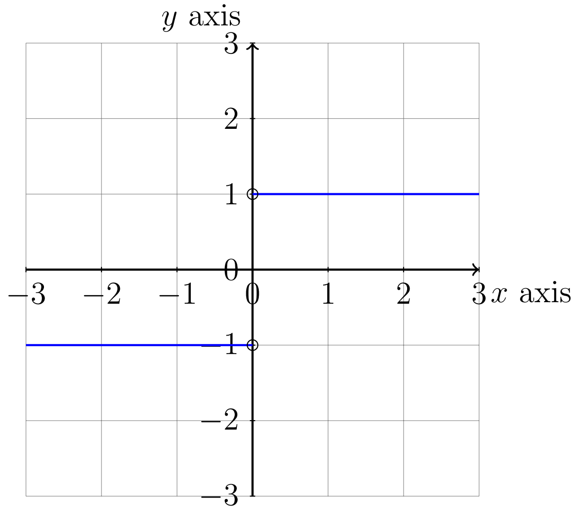
Note that the $\to$ can be read as "*approaches*". You may now start getting a better idea of how we use the idea of *limits* in mathematics. Lets see one more example.

Example 5 *Consider* $f(x) = \frac{x}{|x|}$*, which means the domain of the function excludes 0. To see this, observe that* $f(2) = 1$*, while* $f(-3) = -1$*. Indeed, for positive* x*,* $f(x) = 1$*, while for negative* x*,* $f(x) = -1$*. Graphically, the function is depicted in figure 4.*

The legitimate question then is what is the value of $\lim_{x \to 0} f(x)$ *if it exists at all?*

Solution 5 *We know that as long as* x *remains positive, that is on the positive side of the real line,* $x \to 0$*,* $f(x) \to 1$*. Similarly, as* $x \to 0$ *on the negative side of the real line,* $f(x) \to -1$*. However, when* x *is near 0, it is not the case that it is near any one of*

Figure 4: Graph of $y = \frac{x}{|x|}$

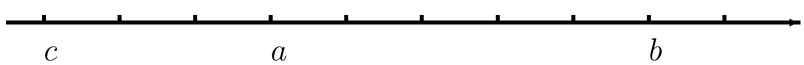


those numbers, -1 or 1, so that $\lim_{x\to 0} \frac{x}{|x|}$ *does not exist. However, for a number "a", if a is positive,* $\lim_{x\to a} \frac{x}{|x|}$ *exists and is equal to 1, and similarly if a is negative,* $\lim_{x\to a} \frac{x}{|x|}$ *is equal to -1.*

The key point of the last example is that whether a function f has a limit has nothing to do with the value of $f(a)$. We can now provide the definition formally.

Definition 8 ***The Limit of*** $f(x)$ ***at*** a**.** *Let* f *be a function, and* a *be a fixed number. Assume that the domain of* f *consists of the interval* (c, a) *and* (a, b) *such as it is described below in figure 5. Suppose there is a number* L *such that as* $x \to a$*, either from the right*

Figure 5:



or left, $f(x) \to L$*, then* L *is called* ***the limit of*** $f(x)$ ***as*** $x \to a$*, and is written as,*

$$\lim_{x\to a} f(x) = L$$
$$\text{or } f(x) \to L \qquad \text{as } x \to a$$

In addition, to ensure there is no confusion, we define what is a right-hand and left-hand limit.

Definition 9 ***Right-hand Limit of*** $f(x)$ ***at*** a. *Let* f *be a function and* a *is some fixed number. Let the domain of* f *include* (a, b). *If as* x *approaches from the right,* $f(x) \to L$, *then* L *is the* ***right-hand limit of*** $f(x)$ ***as*** $x \to a$, *and is written as,*

$$\begin{aligned} \lim_{x \to a^+} f(x) &= L \\ \text{or } f(x) \to L & \quad \text{as } x \to a^+ \end{aligned}$$

This can be read as "the limit of f *as* x *approaches* a *from the right is* L*".*

The ***Left-hand Limit of*** $f(x)$ ***at*** a *can be defined similarly. Let the domain of* f *include* (c, a), *so that we can examine the behavior of the function as* x *approaches from the left. The notation for the left-hand limit is thus,*

$$\begin{aligned} \lim_{x \to a^-} f(x) &= L \\ \text{or } f(x) \to L & \quad \text{as } x \to a^- \end{aligned}$$

It should be noted that at this juncture, these definitions are not precise. To get a precise definition, we need to get towards understanding continuity. But before then, lets see how we can use the "lim" operators to calculate limits of functions. All these may be mathematical curiosities to you at this juncture, until you realize how applicable it is to economics, and life in general. Consider the following question, "if your income is growing at a rate of $x\%$ per annum, what is the growth rate of your annual expenditure that will ensure you have a positive net savings at retirement age?". You should then realize that this question can be translated into a mathematical question of limits, if inflation is kept constant.

2.2 Computing Limits

Before we can start computing limits, it is worth knowing some of the properties in its use.

Theorem 1 *Let* f *and* g *be two functions and assume their limits exists. Then,*

1. $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
2. $k \lim_{x \to a} f(x) = k \lim_{x \to a} f(x)$

3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} \neq 0$

5. $\lim_{x\to a} f(x)^{g(x)} = \left(\lim_{x\to a} f(x)\right)^{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} f(x) > 0$

Lets have some examples to see how these properties are used.

Example 6 *Let* $\lim_{x\to 3} f(x) = 4$ *and* $\lim_{x\to 3} g(x) = 5$, *then what is* $\lim_{x\to 3} \frac{f(x)}{g(x)}$.

Solution 6 *It is clear then by property 4 of theorem 1 that we have* $\lim_{x\to 3} \frac{f(x)}{g(x)} = \frac{4}{5}$.

Consider another,

Example 7 *What is* $\lim_{x\to\infty} \frac{x^3+6x^2+10x+2}{2x^3+x^2+5}$ *?*

Solution 7 *We have,*

$$\begin{aligned} f(x) &= \frac{x^3+6x^2+10x+2}{2x^3+x^2+5} \\ &= \frac{x^3\left(1+\frac{6}{x}+\frac{10}{x^2}+\frac{2}{x^3}\right)}{x^3\left(2+\frac{1}{x}+\frac{5}{x^3}\right)} \\ &= \frac{1+\frac{6}{x}+\frac{10}{x^2}+\frac{2}{x^3}}{2+\frac{1}{x}+\frac{5}{x^3}} \end{aligned}$$

for $x \neq 0$. *It is clear that* $x \to \infty$, $\frac{6}{x} \to 0$, $\frac{10}{x^2} \to 0$, $\frac{2}{x^3} \to 0$, $\frac{1}{x} \to 0$, *and* $\frac{5}{x^3} \to 0$. $\therefore$

$$\lim_{x\to\infty} \frac{x^3+6x^2+10x+2}{2x^3+x^2+5} = \frac{1}{2}$$

The following theorems will formalize further how you calculate a limit to an actual function.

Theorem 2 1. *If* $y = ax + b$, *then* $\lim_{x\to n} y = an + b$, *where* $a, b \in \mathbb{R}$ *are constants.*

2. *If* $y = b$, *where* $b \in \mathbb{R}$ *is a constant, then* $\lim_{x\to n} y = b$.

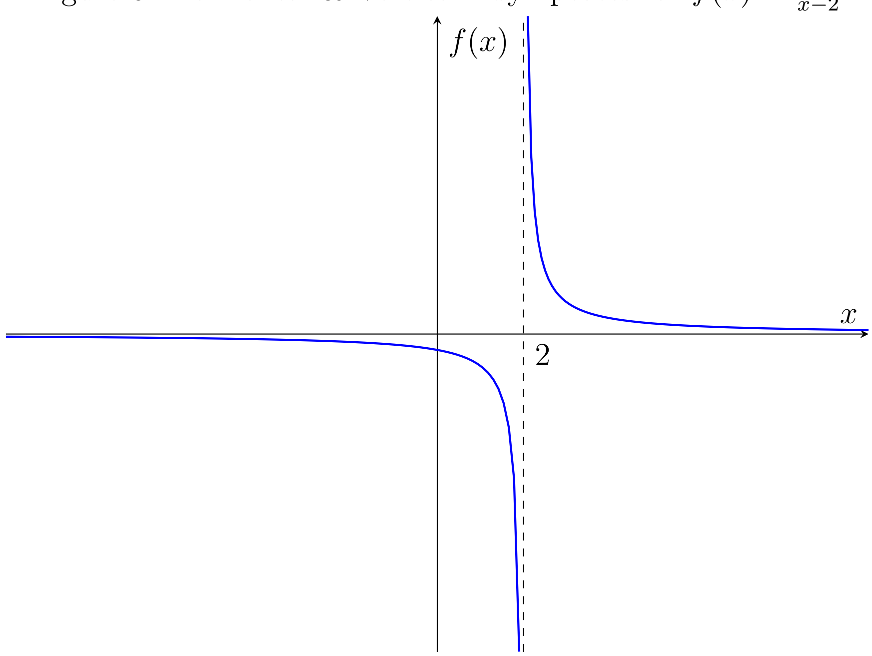
3. *If* $y = x$, *then* $\lim_{x\to n} y = n$. *Further, if* $y = x^k$, *then* $\lim_{x\to n} y = n^k$.

2.3 Asymptotes & their Graph

What happens when x becomes very large, such as in the case of $\lim_{x\to\infty} f(x) = L$, for $L \in \mathbb{R}$? This means that as x gets arbitrarily large, $y = f(x)$ would get arbitrarily close to $y = L$, but never quite getting there. This line $y = L$ is known as a *horizontal asymptote* of the graph of f. This idea works similarly show x becomes arbitrarily small, such as $\lim_{x\to-\infty} f(x) = L$.

On the other hand, if $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = \infty$, the asymptote is a vertical line at $x = a$ since the function never touches $x = a$. You may define a similar vertical asymptote where as $x \to a^+$ or $x \to a^-$ the function becomes very small. That is if $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = -\infty$. From the figure 6 of $f(x) = \frac{1}{x-2}$, we see that there is a horizontal asymptote described by the $x-$axis, that is at $y = 0$. There is also a vertical asymptote at $x = 2$, noting that $\lim_{x\to 2^+} \frac{1}{x-2} = \infty$ and $\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$, an $\lim_{x\to\infty} \frac{1}{x-2} = 0 = \lim_{x\to-\infty} \frac{1}{x-2}$.

Figure 6: Horizontal & Vertical Asymptotes for $f(x) = \frac{1}{x-2}$



You should not however go away thinking that asymptotes need to be solely vertical or horizontal. For example, in figure 7, there are two asymptotes, one at the $y-$axis, that is $x = 0$, and the other that is tilted described by $y = x$. Or yet another is in figure 7, where the asymptote is a curve described by $y = x^2$.

Figure 7: Asymptotes for $f(x) = x + \frac{1}{x}$

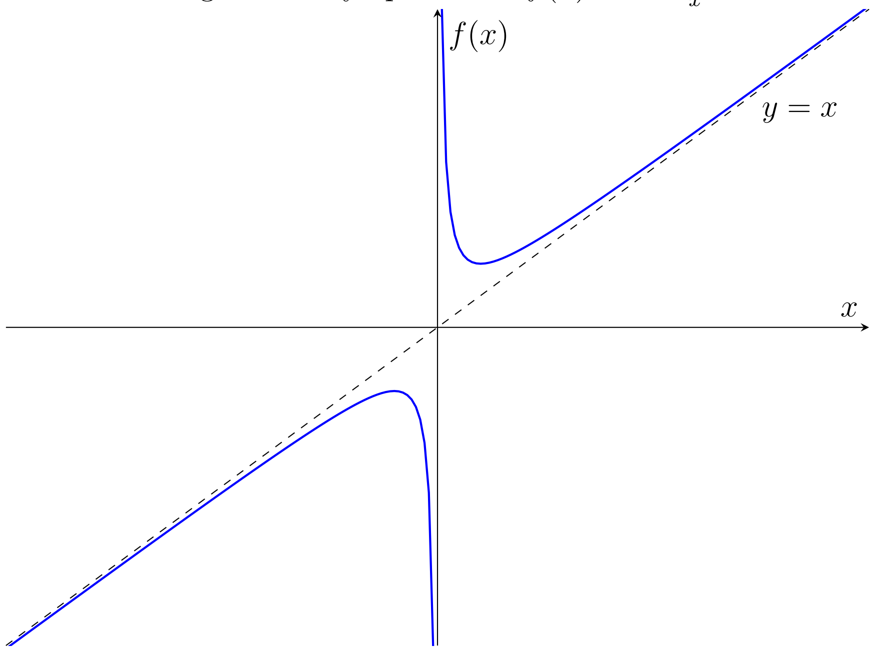
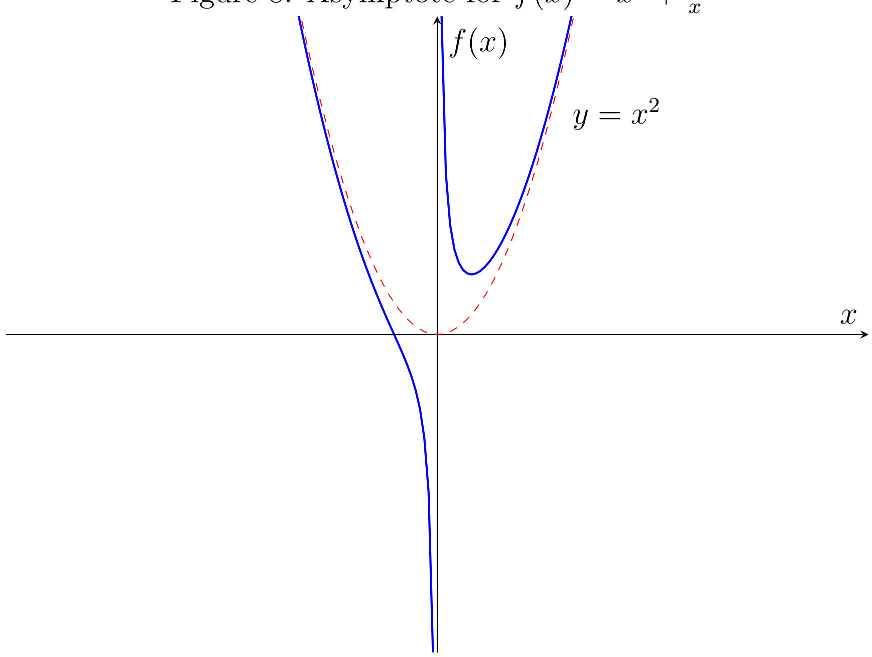


Figure 8: Asymptote for $f(x) = x^2 + \frac{1}{x}$



2.4 Continuous Functions

With the preamble out of the way, we can come to properly describing *continuity* which is important in differential calculus. To get a handle of some kind of intuition about continuity, lets consider an abstract function f with the following values near 1,

x	0.9	0.99	0.999
f(x)	3.93	3.996	3.9999998

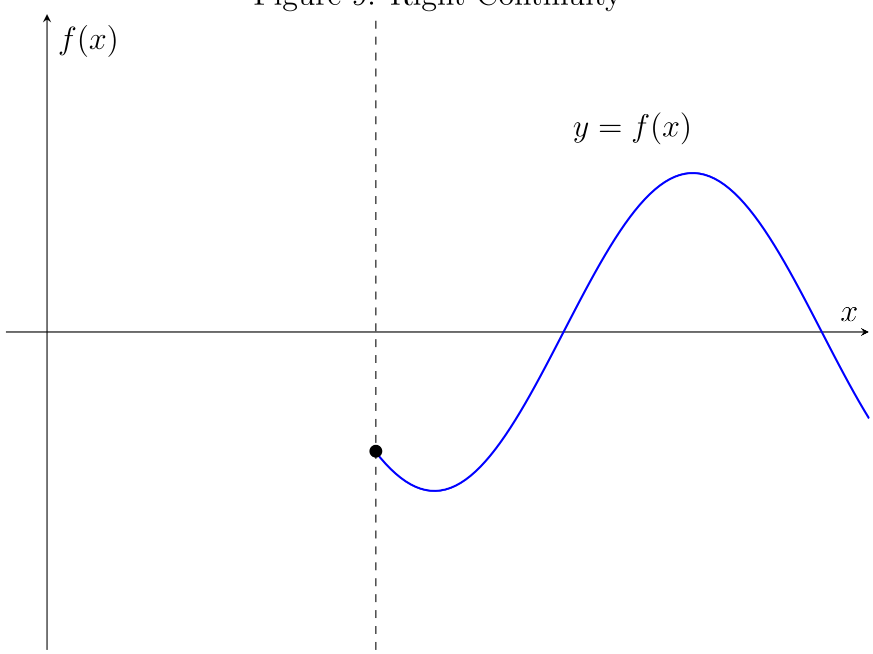
Given the values above, we would expect $f(1)$ to be a number close to 4, as opposed to say -6. Generally speaking, we expect the value of a function at a particular independent variable, say a, to be closely connected to the output at input values close to a. We do not expect sudden jumps. Graphically, we expect the function to be smooth lines or curves, as opposed to a scattering of random points. In other words, we expect functions to be *continuous*. This idea is formalized by the following three definitions.

Definition 10 ***Continuity from the right at a number*** a. *Let* $f(x)$ *be defined at* a*, and on some open interval* (a, b). *Then the function* f *is continuous at* a *from the right if* $\lim_{x \to a^+} f(x) = f(a)$. *That is,*

1. $\lim_{x \to a^+} f(x)$ *exists, and*
2. *the limit is* $f(a)$.

This definition is illustrated in figure 9

Figure 9: Right Continuity

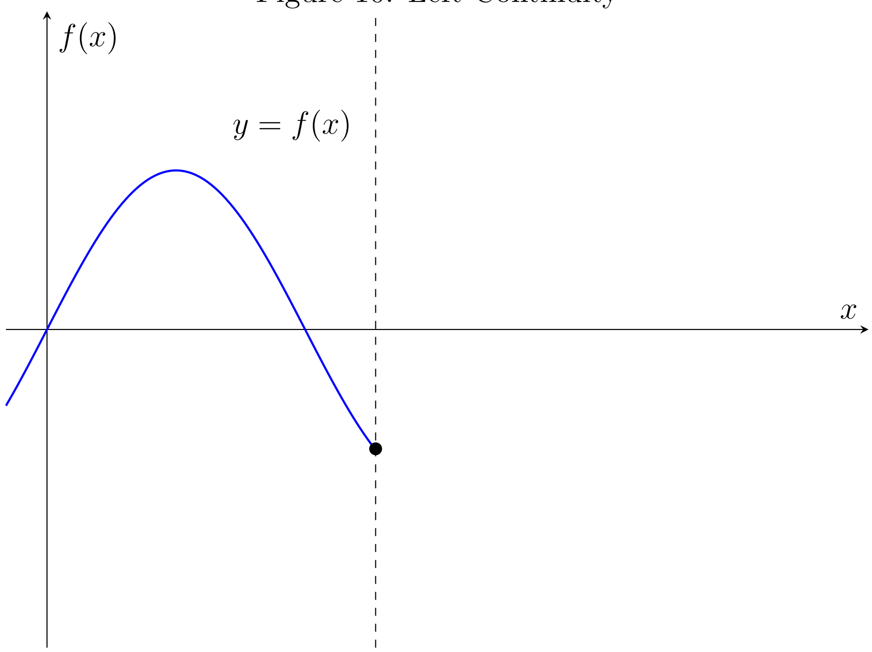


Definition 11 ***Continuity from the left at a number*** *a. Let $f(x)$ be defined at a, and on some open interval (c, a). Then the function f is continuous at a from the left if $\lim_{x \to a^-} f(x) = f(a)$. That is,*

1. *$\lim_{x \to a^-} f(x)$ exists, and*
2. *the limit is $f(a)$.*

This definition is in turn illustrated in figure 10

Figure 10: Left Continuity



Then continuity itself is a combination to the two latter definitions.

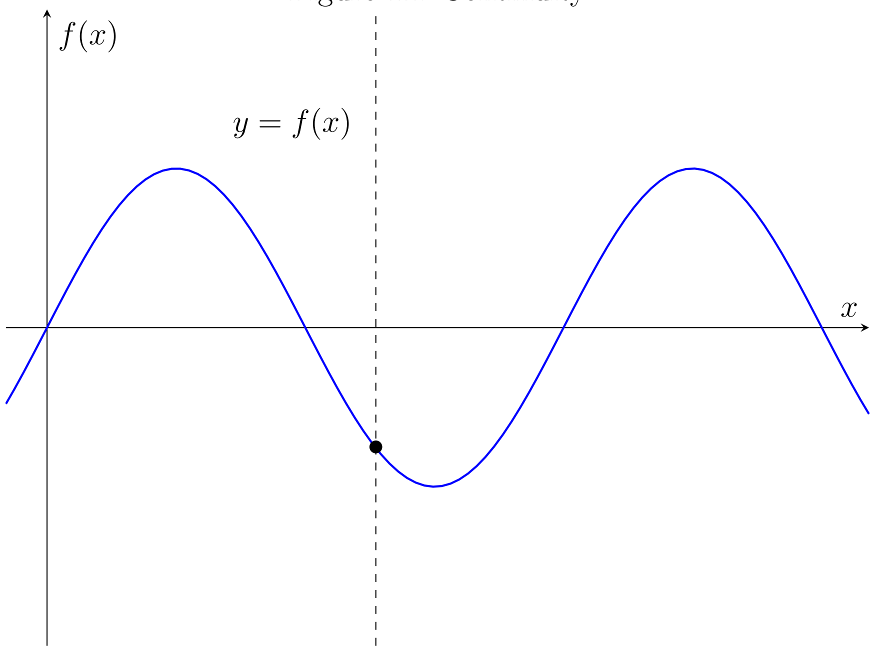
Definition 12 ***Continuity at a number*** *a. Let $f(x)$ be defined at a, and on some open interval (c, b) that contains a, that is $a \in (c, b)$. Then the function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. That is,*

1. *$\lim_{x \to a} f(x)$ exists, and*
2. *the limit is $f(a)$.*

And combining the latter two diagrams, we have,

What has just transpired is the definition of the notion of continuity. But what we need are *continuous functions*. Now what is the definiton of that?

Figure 11: Continuity



Definition 13 ***Continuous Function.*** *Let f be a function whose input domain is the entire real line, or is made up of open intervals on the real line. Then f is a* ***continuous function*** *if it is continuous at each number a in its domain.*

What if we are dealing with closed intervals, such as $[c, b]$? Then to effect the same idea, we need the following modifications. *A function f is continuous on $[c, b]$, if it is continuous at each point in (c, b), and is continuous from the right at c, and is continuous from the left at b.*

How about a half closed and half open interval such as $[a, \infty)$. Well, that modification would read as, *A function f is continuous on $[a, \infty)$, if it is continuous at each point in (a, ∞), and is right continuous at a.* Complete this exercise by providing the definition on the interval $(-\infty, b]$.

Much of the functions we have in mathematics and statistics, and as used in economics are continuous, and their algebraic combinations are likewise continuous. To provide a more formal definition.

Definition 14 ***Sum, Difference, Product, and Quotient of Functions.*** *Let f and g be two functions. The functions $f \pm g$, $f.g$, and $\frac{f}{g}$ are defined as:*

$$
\begin{array}{ll}
(f \pm g)(x) = f(x) \pm g(x) & \textit{for } x \textit{ in the domain of both } f \textit{ and } g \\
(f.g)(x) = f(x).g(x) & \textit{for } x \textit{ in the domain of both } f \textit{ and } g \\
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} & \textit{for } x \textit{ in the domain of both } f \textit{ and } g,\ g(x) \neq 0
\end{array}
$$

2.5 The Maximum-Value Theorem & the Intermediate-Value Theorem

Theorem 3 ***Maximum-Value and Minimum-Value Theorem.*** *Let f be continuous on the interval $[a, b]$. Then there is at least one number in $[a, b]$ at which f will achieve its maximum value. In other words, for some $c \in [a, b]$,*

$$f(c) \geq f(x) \qquad \forall x \in [a, b]$$

Similarly, f will take a minimum value somewhere in the interval.

Given the extremes, and our understanding of a continuous function, it stands to reason that an intermediate value between the extremes must also lie in the interval, which leads on to the following theorem.

Theorem 4 ***Intermediate-Value Theorem.*** *Let f be continuous on $[a, b]$. Let m be such that $f(a) \leq m \leq f(b)$ if $f(a) \leq f(b)$, or alternatively $f(a) \geq m \geq f(b)$ if $f(a) \geq f(b)$. Then there is at least one number $c \in [a, b]$ such that $f(c) = m$.*

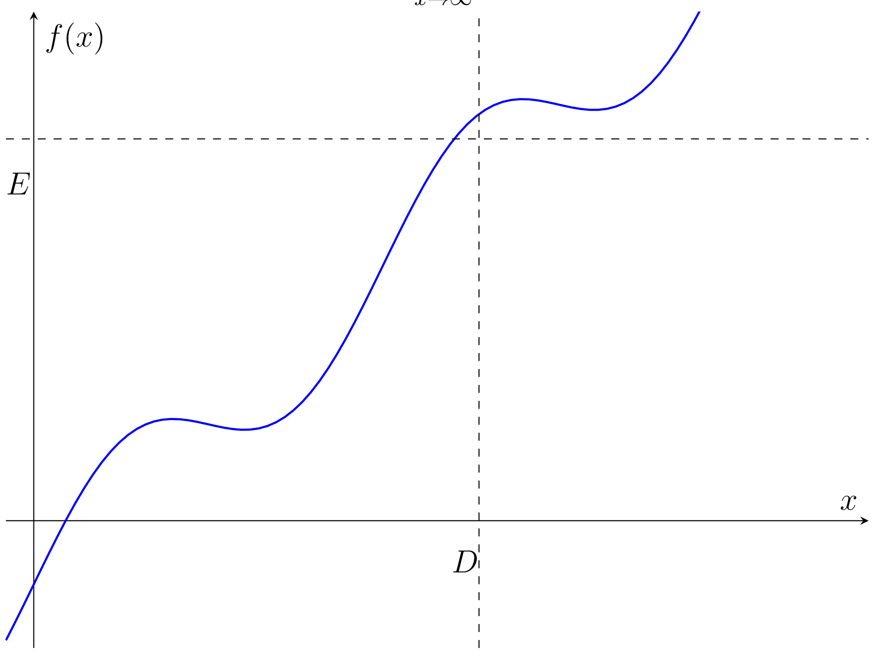
2.6 Precise Definitions

There that your familiarity with mathematical notation, and terminology is improving, we can get more precise about some of the key definitions regarding limits to end this chapter.

Definition 15 *Assume that $f(x)$ is defined $\forall x > c$, where c is some arbitrary number within the domain. Then $\lim_{x \to \infty} f(x) = \infty$ means that for each number E, there is a number D such that $\forall x > D$, it is trye that $f(x) > E$.*

To get a better idea of that the definition is saying, lets focus on figure 12. E is a "challenge" to the idea, that the larger E is, the larger must D be. Only if we can find a number D, which in turn depends on E, for every E, can the claim $\lim_{x \to \infty} f(x) = \infty$ be true. Based on the diagram, for each possible horizontal line denoting $y = E$, if we are to extend far to the right of the graph of function f, the function would stay above $y = E$. In other words, we can find a D, such that for $x > D$, then $f(x) > E$. Keep in mind that we are talking about one-to-one functions. Further, the "response" D is not unique.

Figure 12: $\lim_{x\to\infty} f(x) = \infty$



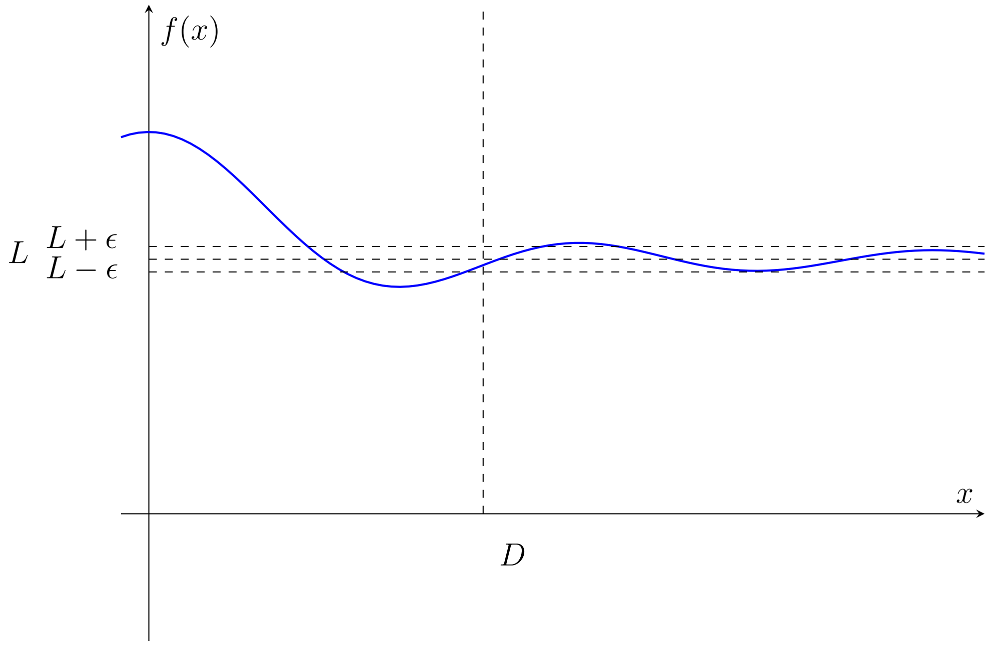
Another idea about limits, $\lim_{x\to\infty} f(x) = L$ which we can stated informally, can now be formalized.

Definition 16 *Assume that $f(x)$ is defined $\forall x > c$, where c is some arbitrary number within the domain. For each positive number, ϵ, there is a number D such that $\forall x > D$, it it true that*

$$|f(x) - L| \quad < \quad \epsilon$$

Similar to the previous discussion, ϵ is a challenge to discovery, and D is the response. So the smaller ϵ is, the smaller the band becomes, such as in figure 13. In consequence, the larger D needs to be. By narrowing the band, we will begin to find the limit then.

Figure 13: $\lim_{x\to\infty} f(x) = L$



The following example shows how this idea could be used.

Example 8 *Using the definition of "*$\lim_{x\to\infty} f(x) = L$*", show that*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$$

Solution 8 *It may be seen that* $f(x) = \left(1+\frac{1}{x}\right)$ *is defined for all* $x > 0$*, and that the definition's* $L = 1$ *here. We need to show that for any positive* ϵ *(since we are examining* $x \to \infty$*), however small, we would still be able to find* D*, such that* $\forall x > D$

$$\left|\left(1+\frac{1}{x}\right)-1\right| < \epsilon$$

which reduces to,

$$\left|\frac{1}{x}\right| < \epsilon$$

and further since we are dealing with $x > 0$*, we have*

$$\begin{aligned} \frac{1}{x} &< \epsilon \\ \Rightarrow 1 &< \epsilon x \\ \Rightarrow x &> \frac{1}{\epsilon} \end{aligned}$$

so that setting $D = \frac{1}{\epsilon}$, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$ *will be true.*

The last precise definition pertains to the limit taken to a fixed number, as opposed to the abstract ∞.

Definition 17 *Assume that* $f(x)$ *is defined in some intervals* $x \in (c, a)$ *and* $x \in (a, b)$. *For each positive number,* ϵ, *there is a positive number* δ *such that* $\forall x$ *that satisfies the inequality*

$$0 < \quad |x - a| \quad < \delta$$

it must be true that,

$$|f(x) - L| \quad < \quad \epsilon$$

These ideas could be better understood graphically from figure 14. First, $|x - a| > 0$ should be interpreted literally, that is, it says x is not a. While $|x - a| < \delta$ says that x is within a distance of δ from a. a is purely some arbitrary reference point, and what is important is the interval generated of $(a - \delta, a + \delta)$. As in the previous definitions, ϵ is a challenge to the statement, while the response is δ. The smaller ϵ is, the smaller δ needs to be, in order to obtain the limiting value of $f(x)$. To see that latter point, from the diagram, observe that for δ' which generates a wider interval in response to ϵ, $|f(x) - L| < \epsilon$ is not true.

Figure 14: $\lim_{x \to \infty} f(x) = L$