Quantitative Methods in Economics
ECON 271:10
Chapter 4
The Definite Integral

Teng Wah Leo

What we have done till now relates to local information such as gradients and velocity. However, calculus is powerful because it also provides us with global information which we will discuss now. It is the process of finding the antiderivative, and the process is known as *integration*, or finding the *integral*. We will first motivate the problem.

1 An Area Problem

Suppose you wish to find the area under a curve, $y = x^2$ such as in figure 1 for domain $x \in [0, 10]$. An estimate could be made using a set of 10 histograms such as in figure 2. This would give us a total of 10 histograms centered on each of the integers (except at 0, and 10),

$$1^2(1) + 2^2(1) + 3^2(1) + \cdots + 10^2(.5)$$

which gives us 335. However, as may be seen in figure 2, the area we have calculated is inaccurate because it is only an approximation. It is clear that the histogram is above some part of the curve, and below others, which might get you wondering, don't they canceled out. But how do you know it will? But the idea is moving in the right direction. What if we start making the histogram thinner, and include more of them. This should gradually negate the bias. Now, if you think about it, does not this seem like using the idea of limits?

Figure 1: Area Problem for $y = x^2$

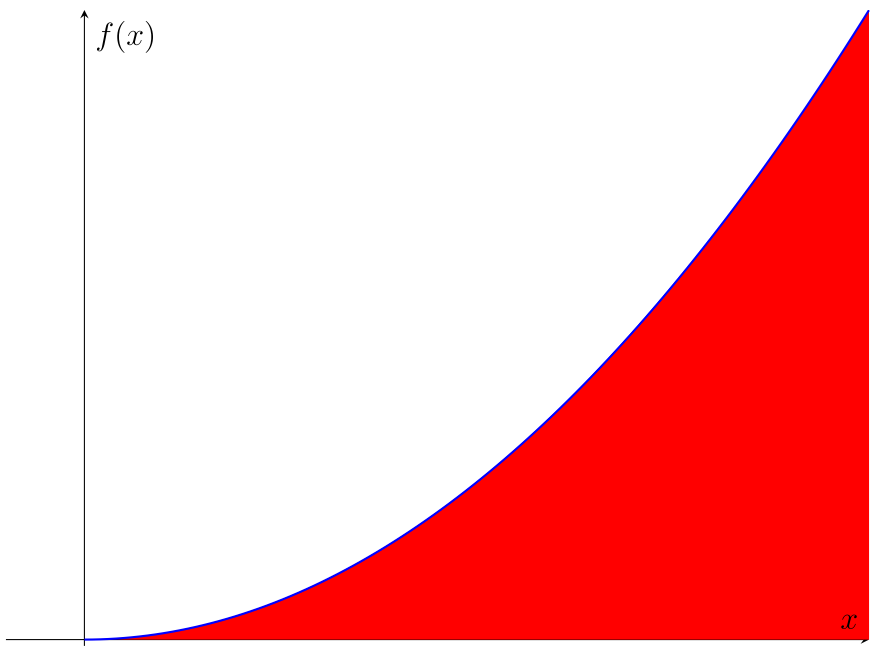
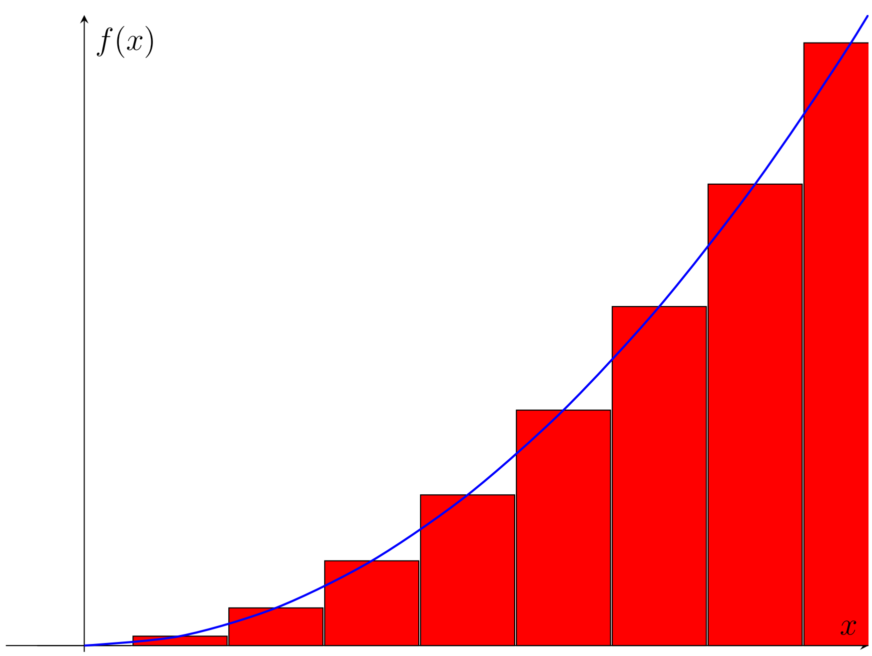


Figure 2: Approximating the Area of $y = x^2$

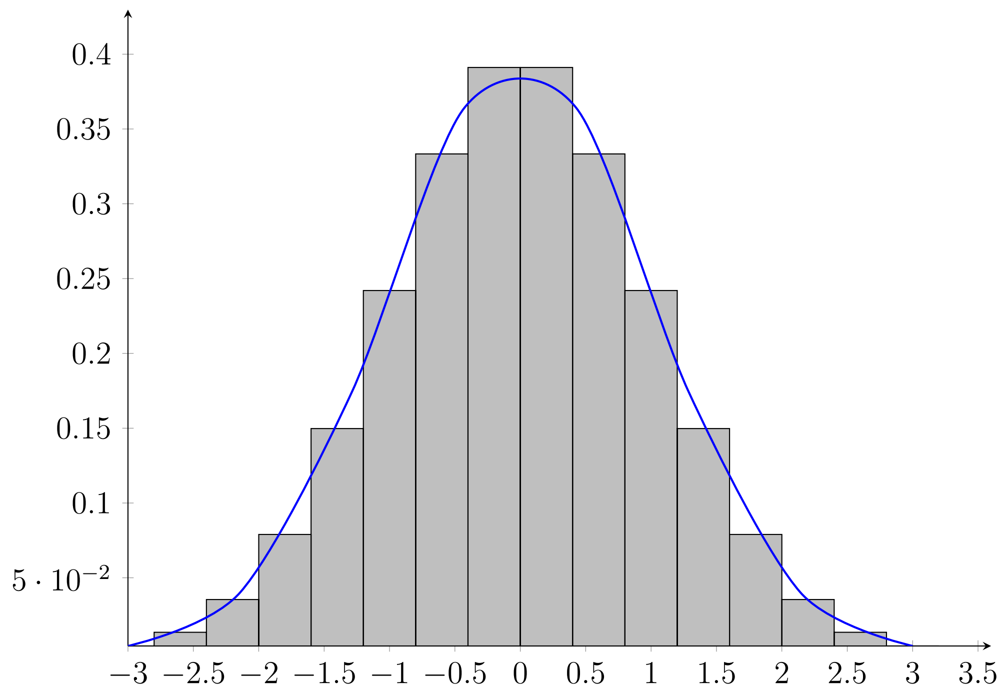


You might also be wondering when would we need to know the area? Let us consider the much used distribution you often think about, and use, the Gaussian distribution, or more colloqially, the bell curve. It has an equation of,

$$\phi(x) = \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

which is a continuous function. So that the probability at any point on the domain of $x \in \mathbb{R}$ is zero. How do you find the probability of an event then? Well, you can calculate it for a subset of the domain, which is then the *area* underneath the distribution, such as in 3!

Figure 3: Approximating the Area of the Gaussian Distribution



Indeed, this idea extends to distance, and volume problems as well.

2 Summation Notation & Approximating Sums

Since the approximations involves our first considering sums, and since the degree of accuracy increases with finer and finer dicing of the domain, and in consequence larger sums, we introduce this notation here now.

Definition 1 ***Summation Notation***. *Let $b_1, b_2, \ldots, b_n$ be n numbers, whose sum may be denoted using the sigma notation, $\sum$, that is,*

$$\sum_{i=1}^{n} b_i = b_1 + b_2 + \cdots + b_n$$

which is read as "the sum of b_i as i goes from 1 to n".

Consider the following example, to see how this might be used.

Example 1 *Write the sum of $1^2 + 2^2 + 3^2 + 4^2$ in the summation notation.*

Solution 1

$$\sum_{i=1}^{n} i^2 = 30$$

Or the following.

Example 2 *Compute $\sum\limits_{k=1}^{4} \frac{1}{k}$*

Solution 2

$$\sum_{k=1}^{4} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

It is important to note that the subscript tells you the variable that is changing in the summation. So that if we multiple a constant, a number that does not change, to each of the element in the sum, we can factor it out of the sum. In other words,

$$\begin{aligned}\sum_{i=n}^{n} cb_i &= cb_1 + cb_2 + \cdots + cb_n \\ &= c(b_1 + b_2 + \cdots + b_n) = c\sum_{i=1}^{n} b_i\end{aligned}$$

This is often read as "a constant factor can be moved past $\sum$".

It is clear that the standard rules of algebra applies. The next rule says that you can always group variables together in their own sums, which is drawn from the distributive rule in algebra. In other word,

$$\begin{aligned}\sum_{k=1}^{n}(b_k + c_k) &= (b_1 + c_1) + (b_2 + c_2) + \cdots + (b_n + c_n) \\ &= (b_1 + b_2 + \cdots + b_n) + (c_1 + c_2 + \cdots + c_n) \\ &= \sum_{k=1}^{n}(b_k) + \sum_{k=1}^{n}(c_k)\end{aligned}$$

Example 3 *Compute* $\sum_{i=1}^{4}\left(i+\frac{1}{i}\right)$.

Solution 3

$$\sum_{i=1}^{4}\left(i+\frac{1}{i}\right)=\sum_{i=1}^{4} i+\sum_{i=1}^{4}\frac{1}{i}=30+2.083=32.083$$

What if you are asked to sum a constant n times? In other words,

$$\sum_{i=1} c = c+c+\cdots+c = nc$$

So that as you should have intuitively expected, it means you multiple n to the constant.

What if the index does not begin with 1? Well, what is wrong with that. Consider the following example.

Example 4 *Compute* $\sum_{k=2}^{6} 5k$.

Solution 4

$$\sum_{k=2}^{6} 5k \quad = \quad 5(2+3+4+5+6)=100$$

Sometimes, it might be useful, and possible to split up, or combine the summation. Consider the following,

$$\sum_{j=1}^{m} b_j+\sum_{j=m+1}^{n} b_j=\sum_{j=1}^{n} b_j$$

You might wonder how this might be useful? Consider the following example.

Example 5 *Let* b_0, b_1, b_2 *and* b_3 *be four numbers. Form the three differences,* $a_1=b_1-b_0$, $a_2=b_2-b_1$, $a_3=b_3-b_2$, *and compute* $\sum_{i=1}^{3} a_i$.

Solution 5

$$\sum_{i=1}^{3} a_i \quad = \quad (b_1-b_0)+(b_2-b_1)+(b_3-b_2)=b_3-b_0$$

This is an example of a *telescoping sum*, $\sum_{i=1}^{n}(b_i-b_{i-1})$ which "telescopes" to b_n-b_0. This is useful in obtaining formulae for possibly elaborate sums, such as $1+2+\cdots+n$

and $1^2+2^2+\cdots+n^2$. Consider $\sum_{i=1}^{n}[(i+1)^2-i^2]$. We know based on our previous example, the sum is equal to $(n+1)^2-1$. But, based on the Binomial Theorem,

$$(i+1)^2 - i^2 \quad = \quad 2i+1$$

which means thus that,

$$\begin{aligned}
\sum_{i=1}^{n}(2i+1) &= (n+1)^2-1 \\
\Rightarrow 2\sum_{i=1}^{n} i + n &= n^2+2n \\
\Rightarrow \sum_{i=1}^{n} i &= \frac{n^2+n}{2} = \frac{n(n+1)}{2}
\end{aligned}$$

Let us look at finite sum of squares, starting with,

$$\sum_{i=1}^{n}[(i+1)^3-i^3] \quad = \quad (n+1)^3-1$$

which again, by the Binomial Theorem,

$$(i+1)^3 \quad = \quad i^3+3i^2+3i+1$$

This thus means that,

$$\begin{aligned}
\sum_{i=1}^{n}[i^3+3i^2+3i+1-i^3] &= n^3+3n^2+3n \\
\Rightarrow \sum_{i=1}^{n}[3i^2+3i+1] &= n^3+3n^2+3n \\
\Rightarrow 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n &= n^3+3n^2+3n \\
\Rightarrow 3\sum_{i=1}^{n} i^2 + \frac{3n(n+1)}{2} + n &= n^3+3n^2+3n \\
\Rightarrow 3\sum_{i=1}^{n} i^2 + \frac{3n^2}{2} + \frac{5n}{2} &= n^3+3n^2+3n \\
\Rightarrow 3\sum_{i=1}^{n} i^2 &= n^3 + \frac{3n^2}{2} + \frac{n}{2} \\
\Rightarrow \sum_{i=1}^{n} i^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

At this juncture, you might still be wondering what was all this discussion for. How has this got to do with finding or approximating areas?

Consider the area under the function which we had discussed earlier. What we did was that we sliced and diced the domain into partitions, and created bars approximating the area under the function. Let us do the same now. Let the domain under a function, say $f(x)$ be $[\underline{x}, \overline{x}]$. Let us cut the domain into its $n-1$ partitions, such that $\underline{x} = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \overline{x}$. Denote the mass at the ith partition, which may be measured at x_{i-1} or x_i, or $\frac{x_{i-1}+x_i}{2}$, as c_i. This number c_i is known as the *sampling number*. Then the estimate of the areas can be written as,

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \;\; = \;\; \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This is known as a *Riemann sum* after the nineteenth century mathematician, Georg Riemann. Now we are ready to discuss integrals.

3 Definite Integrals

The idea behind finding global features of a function, such as area, is very similar to that of derivatives. Looking at the Riemann sum, you would realize that our estimate of the area gains in accuracy as the partitions chosen increases in number, that is as the distance between the edges of the partitions fall, i.e. becomes more fine. Before we continue with our discussion, we need some definitions for our terminology.

Definition 2 ***Mesh***. *The mesh of a partition is the length of the longest section (sections) in the partition.*

For example, in figure 3, with the Gaussian distribution, the mesh is $\frac{1}{2}$.

Definition 3 ***The definite integral of a function f over a closed interval,*** $[a, b]$. *For a function f defined on $[a, b]$, and with $\sum_{i=1}^{n} f(c_i)\Delta x_i$ approaching a particular number as the mesh of the partitions shrinks towards 0, regardless of how the sampling number c_i is obtained, that number is known as the* ***definite integral of f over*** $[a.b]$. *It is denoted as,*

$$\int_a^b f(x)dx$$

It is also known as the ***Riemann integral****.*

The symbol $\int$ is derived from the letter s associated with the sum, and dx traditional suggests small sections of the domain of f. We will not prove existence of an integral, but state the theorem here. If you are interested, you should enrol into advanced mathematics courses.

Theorem 1 ***Existence of the Definite Integral***. *For a continuous function defined on the interval $[a, b]$, the approximating sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ approaches 0 as the mesh of the partitions of $[a, b]$ tends towards 0. Thus $\int_a^b f(x)dx$ exists.*

Although we had couched the discussion on area, it should not be construed that integration is used solely for that purpose, since it is often used for evaluating volume, mass, and distances as well.

3.1 The Fundamental Theorems of Calculus

We will now examine a primary relationship in calculus, which provides the tools for calculating integrals, without having to formulate an approximating sum, through understanding the relationship between the definite integral and the derivative.

Theorem 2 ***First Fundamental Theorem of Calculus***. *For a continuous function f defined on $[a, b]$, where F is the antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. Denote $G(x) = \int_a^x f(t)dt$. By the second fundamental theorem (below), $G'(x) = f(x)$. Since we have assumed that $F'(x) = f(x)$, then F and G have the same derivative, and all that differs between them is a constant.

$$\begin{aligned} G(x) &= F(x) + C \\ \Rightarrow G(b) - G(a) &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) \end{aligned}$$

But we know,

$$\begin{aligned} G(a) &= \int_a^a f(x)dx = 0 \\ \& \ G(b) &= \int_a^b f(x)dx \\ \Rightarrow G(b) - G(a) &= \int_a^b f(x)dx \end{aligned}$$

Therefore,

$$\int_a^b f(x)dx = G(b) - G(a) = F(b) - F(a)$$

and the result follows. ■

The above theorem says thus that should you wish to find $\int\limits_a^b f(x)dx$, all you need to do is to find the antiderivative of $f = \Delta F = F'$.

Example 6 *Compute* $\int\limits_a^b x^2 dx$.

Solution 6 *Following the first Fundamental Theorem of Calculus, and realizing that differentiating* $\frac{x^3}{3}$ *gives you* x^2*, so that* $F(x) = \frac{x^3}{3}$

$$\int\limits_a^b x^2 dx = F(b) - F(a) = \frac{b^3}{3} - \frac{a^3}{3}$$

There is yet another fundamental theorem of calculus that since the definite integral in a different light. Suppose f is a continuous function such that it is positive on the interval $[a, b]$ (this is not necessary, but is easier for our discussion). For $x \in [a, b]$, let $G(x)$ be the area under the graph of f, and above the interval along the $x-$axis from $[a, x]$. Let Δx be an arbitrarily small positive number (such as the distance of a mesh). Then $G(x + \Delta x)$ is the area under the graph of f, and above the interval $[a, x + \Delta x]$. Then $\Delta G = G(x + \Delta x) - G(x)$ is a thin strip of area below the graph of f and above the small interval $[x, x + \Delta x]$. This narrow strip of area would look like a thin rectangle with base of Δx and height $f(x)$, so that the area is $f(x)\Delta x$. So that as $\Delta x \to 0$,

$$\begin{aligned}
\Delta G = G(x+\Delta x) - G(x) &= f(x)\Delta x \\
\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} &= f(x) \\
\Rightarrow \frac{dG(x)}{dx} &= f(x) \\
\Rightarrow G(x) &= \int\limits_a^x f(t)dt
\end{aligned}$$

So that it is not unreasonable to think that

$$\frac{d}{dx}\int\limits_a^x f(t)dt = f(x)$$

holds in general, which leads us to the second fundamental theorem of calculus.

Theorem 3 ***Second Fundamental Theorem of Calculus***. *For a continuous f on an interval $[a, b]$, let*

$$G(x) = \int\limits_a^x f(t)dt$$

where $x \in [a, b]$. Then G is differentiable on $[a, b]$, and its antiderivative is f, in other words, $G'(x) = f(x)$

Proof. Recall that,

$$\frac{\Delta G}{\Delta x} = \frac{G(x+\Delta x) - G(x)}{\Delta x}$$

as $\Delta x \to 0$ for a fixed x. Further, we know Δx can be positive or negative. Next, we know that,

$$\Delta G = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt$$

Further,

$$\int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt$$

Thus,

$$\begin{aligned}\Delta G &= \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+\Delta x} f(t)dt\end{aligned}$$

Based on the mean-value theorem for definite integrals (to follow),

$$\begin{aligned}\Delta G = \int_x^{x+\Delta x} f(t)dt &= f(c)((x+\Delta x) - x) \\ &= f(c)\Delta x \\ \therefore\Rightarrow \frac{\Delta G}{\Delta x} &= f(c)\end{aligned}$$

for some $c \in [x, x+\Delta x]$ or $c \in [x+\Delta x, x]$. Thus,

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{\Delta G}{\Delta x} &= \lim_{\Delta x \to 0} f(c) \\ &= f(x)\end{aligned}$$

which shows that G is differentiable, and that $\frac{d}{dx}G = f$.

■

This thus leads to the corollary that every continuous function is derivative of some function.

Corollary 1 *For f defined on the interval $[a, b]$, f is the derivative of some function.*

3.2 Properties of the Antiderivative & the Definite Integral

As you would have surmised from our prior discussions, the antiderivative of f is denoted by $\int f(x)dx$. So if $F' = f$, we can write $F = \int f(x)dx$, and say F is the antiderivative of f. For example, if $3x^2$ is the derivative of x^3, then x^3 is an antiderivative of $3x^2$. Observe that we say x^3 is an antiderivative, suggesting there may be others. You may suggest $x^3 + 2$ is also an antiderivative of $3x^2$, and you would be correct, since the constant term is not known. The following theorem formalizes this fact.

Theorem 4 *If F and G are both antiderivatives of f on an interval $[a, b]$, then there is a constant C such that*

$$F(x) = G(x) + C$$

There are additional properties who should formalize now before we deal in a later chapter, how to compute systematically, when a solution exists, the antiderivative for a function.

Theorem 5 *Let f and g be functions with respective antiderivatives $\int f(x)dx$ and $\int g(x)dx$. Then the following holds:*

1. $\int cf(x)dx = c\int f(x)dx$, *for any constant c.*
2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$.

Proof. For the first point,

$$\begin{aligned}\frac{d}{dx}\left(c\int f(x)dx\right) &= c\frac{d}{dx}\int f(x)dx \\ &= cf(x) \\ &= \frac{d}{dx}\left(\int cf(x)dx\right)\end{aligned}$$

And for the second,

$$\begin{aligned}\frac{d}{dx}\left(\int f(x)dx \pm \int g(x)dx\right) &= \frac{d}{dx}\left(\int f(x)dx\right) \pm \frac{d}{dx}\left(\int g(x)dx\right) \\ &= f(x) \pm g(x) \\ &= \frac{d}{dx}\left(\int (f(x) \pm g(x))dx\right)\end{aligned}$$

■

From your assignments, you would have surmised that there is a formulæfor finding antiderivatives of a power function.

Theorem 6 *Let b be a rational number, not equal to -1. Then,*

$$\int x^b dx = \frac{x^{b+1}}{b+1} + C$$

where C is just a constant.

The following example shows how you can use the formula.

Example 7 *Find $\int (2x^5 - 3x^2 + 4)dx$.*

Solution 7

$$\begin{aligned}\int (2x^5 - 3x^2 + 4)dx &= \int 2x^5 dx - \int 3x^2 dx + \int 4dx \\ &= \frac{x^6}{3} - x^3 + 4x + C\end{aligned}$$

What if you wish to find the area/volume associated with the function. For that we need additional notation first.

Notation 7 *$F(b) - F(a)$ is written as $F(x)|_a^b$.*

Example 8 *Compute $\int_1^2 \frac{1}{x^2}dx$ by the fundamental theorem of calculus.*

Solution 8

$$\begin{aligned}\int_1^2 \frac{1}{x^2}dx &= -\frac{1}{x}\bigg|_1^2 \\ &= -\frac{1}{2} + 1 = \frac{1}{2}\end{aligned}$$

Thus the first fundamental theorem of calculus asserts that,

$$\int_1^2 \frac{1}{x^2}dx = \int \frac{1}{x^2}dx\bigg|_1^2$$

With this we can provide the formal definition of what we have just performed.

Definition 4 ***Integrand**. For the definite integral $\int_a^b f(x)dx$, and in the antiderivative $\int f(x)dx$, $f(x)$ is known as the **integrand**.*

The process of finding the antiderivative, and the examples above is known as *integrating $f(x)$*. To provide the definition of the relationship between the bounds of the definite integral, we have the following.

Definition 5 *The integral from a to b, where $b < a$, is*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Which then implies the following.

Definition 6 *The integral from a to itself, $\int_a^a f(x)dx = 0$.*

At this juncture, we can note some key properties for Definite Integrals. Consider continuous functions f and g, and a constant c.

1. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.
2. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
3. For $f(x) \geq 0\ \forall x \in [a,b]$, where $a < b$, then $\int_a^b f(x)dx \geq 0$
4. For $f(x) \geq g(x)\ \forall x \in [a,b]$, where $a < b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
5. For constants a, b, and c,
$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$
6. For constants m and M, such that $m \leq f(x) \leq M\ \forall x \in [a,b]$, then
$$\begin{aligned} m(b-a) \leq &\ \int_a^b f(x)dx & \leq M(b-a) \text{if } a < b, \\ \&\ m(b-a) \geq &\ \int_a^b f(x)dx & \geq M(b-a) \text{if } a > b, \end{aligned}$$

The above properties then gives rise to the *Mean-Value Theorem for Definite Integrals.*

Theorem 8 ***Mean-Value Theorem for Definite Integrals****. For constants a and b, and a continuous function f defined for $x \in [a,b]$. Then there is a number c between a and b such that,*
$$\int_a^b f(x)dx = f(c)(b-a)$$

Proof. Without lost of generality, let $a < b$. Further, suppose M is the maximum, and m the minimum value of $f(x)$ for $x \in [a,b]$. We know,
$$\begin{aligned} m(b-a) \leq &\ \int_a^b f(x)dx & \leq M(b-a) \\ \Rightarrow m \leq &\ \tfrac{\int_a^b f(x)dx}{b-a} & \leq M \end{aligned}$$

But by the original mean-value theorem for derivatives, we know there must be a number $c \in [a,b]$ such that,
$$f(c) = \frac{\int_a^b f(x)dx}{b-a}$$

and the theorem follows. ■