Quantitative Methods in Economics
ECON 271:10
Chapter 5
Additional Ideas in Differential Calculus
(Exponential & Logarithmic Functions)

Teng Wah Leo

1 Logarithms

Since you began your studies in math, you would have recognized that,

$$2^? = 4$$

reads as "2 raised to what power is equals to 4?". But what you might not have been taught is that the correct way of stating your answer is, "the logarithm of 4 to the base of 2 is ...". So since we know the exponent should be 2, the full reply is "the logarithm of 4 to the base of 2 is 2". The definition of a logarithm is,

Definition 1 ***Logarithm*** *For positive numbers b ($b \neq 1$) and c, and*

$$b^x = c$$

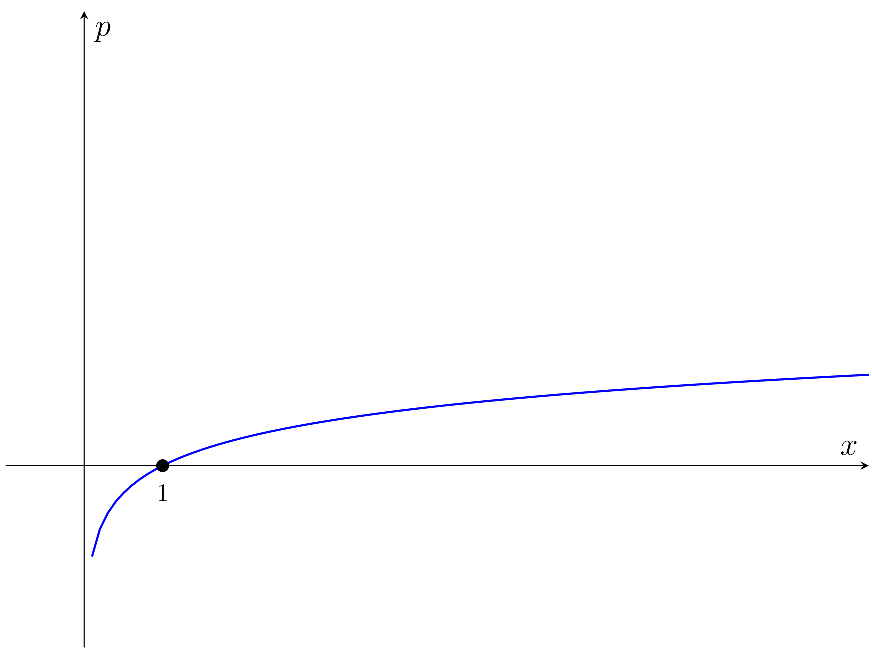
then the number x is the logarithm of c to the base of b, and is written

$$\log_b c$$

Indeed, any *exponential function* $b^x = c$ can be translated into a logarithmic function, $x = \log_b c$.

Logarithms to the base 10, that is $\log_{10} x$ is known as *common logarithms*, and has the following graph.

Figure 1: Graph of $y = \log_{10} x$



Logarithms are defined for $x > 0$, are negative for $x < 1$, and as the base tends towards 1, keeping in mind that the base must be greater than 1. Indeed, for any base $b > 1$, $\lim_{x\to\infty} \log_b x = \infty$ and $\lim_{x\to 0^+} \log_b x = -\infty$.

Like most mathematical operators, logarithms have rules which can be proved. To summarize them quickly,

Exponents	Logarithms
$b^0 = 1$	$\log_b 1 = 0$
$b^{1/2} = \sqrt{b}$	$\log_b \sqrt{b} = \frac{1}{2}$
$b^1 = b$	$\log_b b = 1$
$b^{x+y} = b^x b^y$	$\log_b x + \log_b y$
$b^{-x} = \frac{1}{b^x}$	$\log_b \left(\frac{1}{c}\right) = -\log_b c$
$b^{x-y} = \frac{b^x}{b^y}$	$\log_b \left(\frac{c}{d}\right) = \log_b c - \log_b d$
$(b^x)^y = b^{xy}$	$\log_b c^n = n \log_b c$

Let us try to proof a few of the above that has to be discussed.

Theorem 1 *For any positive numbers c and d, and any base b, $b > 1$,*

$$\log_b cd = \log_b c + \log_b d$$

Proof. By definition, we know $c = b^{\log_b c}$ and $d = b^{\log_b d}$, so that

$$\begin{aligned} cd = b^{\log_b c} b^{\log_b d} &= b^{\log_b c + \log_b d} \\ \Rightarrow \log_b cd &= \log_b b^{\log_b c + \log_b d} \\ &= \log_b c + \log_b d \end{aligned}$$

■

Theorem 2 *For any positive number c, and base $b > 0$,*

$$\log_b \frac{1}{c} = -\log_b c$$

Proof. Let $\log_b c = x$ which implies that $c = b^x$ by definition. Thus $c^{-1} = b^{-x}$.

$$\begin{aligned} c^{-1} &= b^{-x} \\ \Rightarrow \log_b \frac{1}{c} &= -x = -\log_b c \end{aligned}$$

■

Indeed,

Theorem 3 *For any positive number c, base $b > 0$, and real number n,*

$$\log_b c^n = n \log_b c$$

Proof. $\log_b c = x$ which implies that $c = b^x$ by definition. Thus $c^n = (b^x)^n = b^{xn}$.

$$\begin{aligned} c^n &= b^{xn} \\ \log_b c^n &= \log_b b^{xn} \\ &= xn = n \log_b c \end{aligned}$$

■

2 The Number e

Consider our bank deposit concerns. Suppose you have $\$D$ to deposit, and would like to deposit this monies with a bank. Suppose the bank pays r in terest to your deposit, so that your deposit, at the end of the deposit period is $\$(1+r)D$.

Now assume the bank pays 100% in deposit per year. So that for every dollar you deposit, at the end of the year, you get $\$(1+1)D$. If there is another bank realizing the value of your deposit, offers compounding twice a year, so that they pay you 50% per dollar, every six months, you would get $\$\left(1+\frac{1}{2}\right)$ after the first six months, and $\$\left(1+\frac{1}{2}\right)^2 = \$2.25$ after one year. (There is no redepositing of your interest in these calculations.) In other words, the more times your deposit is compounded, the more you deposit grows. To see, suppose yet another competing bank comes and offers compounding three times a year, i.e. every 4 months. This gives you after one year $\$\left(1+\frac{1}{3}\right)^3 \approx \$2.37$. How, about 4 times a year of compounding? You would then get, $\$\left(1+\frac{1}{4}\right)^4 \approx \$2.44$. So that if your interest is deposited n times a year, you get

$$\$\left(1+\frac{1}{n}\right)^n \approx \$2.37$$

. It turns out that this calculation gets you arbitrarily close to a number called e.

Definition 2 *The number e is,*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \approx 2.718281828$$

You may rewrite this expression realizing $\frac{1}{n}$ tends to becoming a very small number as $n \to \infty$, that is,

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

How useful is this idea? How is it related to what we have done till now.

Example 1 *What is* $\lim_{h\to 0}(1+2h)^{1/h}$*?*

Solution 1

$$\begin{aligned}\lim_{h\to 0}(1+2h)^{1/h} &= \lim_{h\to 0}(1+2h)^{/2h} \\ &= \lim_{h\to 0}\left((1+2h)^{1/h}\right)^2 \\ &= e^2\end{aligned}$$

Let's return to our considerations of compounding of interest, but not suppose the interest is more like our current rate of 1.5% a year or per annum, instead of 100%. In that case, we can adjust our formula, for a similar n period compounding to get,

$$\left(1+\frac{0.015}{n}\right)^{n}$$

which based on our last example gives us, $\lim_{n\to\infty}\left(1+\frac{0.015}{n}\right)^{n} = e^{0.015}$. So that continuous compounding yields you approximately \$1.01511, which is just keeping your dollar for 1 year at the bank. You may also generalize that continuous compounding gives you that formula of e^{r}, where r is the interest rate.

It is interesting to note that the number e as found by Leonhard Euler is denoted as,

$$1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$$

3 Derivative of the Logarithmic Functions

We are now able to solve the issue of finding a solution for $\int \frac{1}{x}dx$.

Theorem 4 *The derivative of the function* $\log_b x$ *is,* $\frac{\log_b e}{x}$ *for positive* x.

Proof. The derivative of $\log_b x$ by definition is,

$$\lim_{\Delta x\to 0}\frac{\log_b(x+\Delta x)-\log_b x}{\Delta x}$$

Examining the expression,

$$\begin{aligned}
\frac{\log_b +\Delta x)-\log_b x}{\Delta x} &= \frac{\log_b\left(\frac{x+\Delta x}{x}\right)}{\Delta x} \\
&= \frac{1}{\Delta x}\log_b\left(1+\frac{\Delta x}{x}\right) \\
&= \log_b\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} \\
&= \log_b\left[\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right]^{\frac{1}{x}} \\
&= \frac{1}{x}\log_b\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{1}{x} \log_b \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} &= \frac{1}{x} \lim_{\Delta x \to 0} \log_b \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \\
&= \frac{1}{x} \log_b \left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right] \\
&= \frac{1}{x} \log_b e
\end{aligned}$$

Thus implying $\frac{d}{dx} \log_b x = \frac{\log_b e}{x}$. ∎

Observe that the formula is the simplest when the base $b = e$, since then we have,

$$\frac{d}{dx} \log_e x = \frac{\log_e e}{x} = \frac{1}{x}$$

since $\log_e e = 1$. This is the reason why in calculus, the base e is the preferred base, and we denote $\log_e$ as ln, read as *natural logarithm*. The formula then for its derivative is,

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

for $x > 0$.

This then permits us to discuss the antiderivative of $\frac{1}{x}$.

Theorem 5 *For $x > 0$ or $x < 0$,*

$$\int \frac{1}{x} dx = \ln |x| + c$$

Proof. Since we have shown the derivative of $\ln x$ is $\frac{1}{x}$, we are left to show that the same is true for $\ln |x|$. For $x < 0$, we have,

$$\begin{aligned}
\frac{d}{dx} \ln |x| &= \frac{d}{dx} \ln |-x| \\
&= \frac{1}{-x} \frac{d}{dx}(-x) \\
&= -\frac{1}{x}(-1) = \frac{1}{x}
\end{aligned}$$

so that $\ln |-x|$ has the same antiderivative. ∎

It has to be kept in mind that $\ln x$ or $\log_b x$ is not defined for $x < 0$.

This then leads to a more general idea for integration of functions.

Theorem 6 *For a differentiable function f, and $f(x) \neq 0$,*

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

where C is a constant.

Example 2 *Compute $\int \frac{x^2}{x^3+1} dx$.*

Solution 2 *Although the numerator is not exactly the derivative of the denominator, it is clear that multiplying by a suitable constant makes it so. Precisely,* 3. *Therefore,*

$$\begin{aligned} \int \frac{x^2}{x^3+1} dx &= \frac{1}{3} \int \frac{3x^2}{x^3+1} dx \\ &= \frac{1}{3} \ln|x^3+1| + C \end{aligned}$$

Example 3 *Differentiate $f(x) = \frac{\sqrt[3]{x}\sqrt{(1+x^2)^3}}{x^{\frac{4}{5}}}$.*

Solution 3 *Rather than use the quotient and chain rule in concert, we could apply a* log *transformation to make the process faster, and simpler.*

$$\begin{aligned} \ln f(x) &= \frac{1}{3}\ln x + \frac{3}{2}\ln(1+x^2) - \frac{4}{5}\ln x \\ &= -\frac{7}{15}\ln x + \frac{3}{2}\ln(1+x^2) \\ \Rightarrow \frac{1}{f(x)}\frac{df(x)}{dx} &= -\frac{7}{15x} + \frac{3}{2}\frac{2x}{(1+x^2)} \\ \Rightarrow \frac{df(x)}{dx} &= \left(\frac{\sqrt[3]{x}\sqrt{(1+x^2)^3}}{x^{\frac{4}{5}}}\right)\left(-\frac{7}{15x} + \frac{3}{2}\frac{2x}{(1+x^2)}\right) \end{aligned}$$

4 Derivative of b^x

It should first be noted that since $y = b^x$, then $x = \log_b y$, for $b \neq 1$, so that each function is the inverse of the other. We will use this idea to obtain the derivative of choice, in other words, to find the derivative of the inverse of a function, and there after, obtain the derivative that is desired.

Theorem 7 *For a function f with domain $[a,b]$, and range $[c,d]$. Without loss of generality, let $f' > 0$ $\forall x \in [a,b]$. Then f is a one-to-one function, and its inverse g from $[c,d]$ to $[a,b]$ is differentiable. The same is true if $f' < 0$ $\forall x \in [a,b]$.*

How will this idea be used? Observe that for $y = f(x)$ and $x = g(y)$, we have $x = g(f(x))$, so that $g \circ f$ can be thought of as a composite function, so that by chain rule, we have

$$\begin{aligned} \frac{dx}{dx} &= \frac{dg}{df}\frac{df}{dx} \equiv \frac{dx}{dy}\frac{dy}{dx} \\ \Rightarrow 1 &= g'(f(x))f'(x) \end{aligned}$$

Thus,

Theorem 8 ***The derivative of*** e^x***.*** *Let* $y = e^x$*. Then,*

$$\frac{de^x}{dx} = e^x$$

Proof. We know that for $y = e^x$, we have $x = \ln y$. So that,

$$\begin{aligned} 1 &= \frac{d\ln y}{dx}\frac{dy}{dx} \\ &= \frac{1}{y}\frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= y = e^x \end{aligned}$$

■

Consider now the following examples.

Example 4 *Find the derivative of* e^{4x}*.*

Solution 4 *Let* $y = e^{4x}$*. Define then* $y = e^u$ *for* $u = 4x$*. Then applying chain rule,*

$$\begin{aligned} \frac{de^{4x}}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{de^u}{du}\frac{d(4x)}{dx} \\ &= 4e^u = 4e^{4x} \end{aligned}$$

Example 5 *Find the derivative of* 10^x*.*

Solution 5 *First denote,*

$$\begin{aligned} 10 &= e^{\ln 10} \\ \Rightarrow 10^x &= \left(e^{\ln 10}\right)^x \\ &= e^{(\ln 10)x} \end{aligned}$$

Now we denote, $y = e^{(\ln 10)x}$*, which as before, we can write as* $y = e^u$ *where* $u = (\ln 10)x$*. Therefore,*

$$\begin{aligned} \frac{dy}{dx} &= \frac{de^u}{du}\frac{du}{dx} \\ &= e^u(\ln 10) \\ &= 10^x \ln 10 \end{aligned}$$

This thus provides us with the formula for derivatives of any base b,

$$\frac{d(b^x)}{dx} = (\ln b)b^x$$

With our discoveries, we can now generalize the power function rule to not only rational, but all real numbers a.

Theorem 9 *Let* a *be a real numbered constant. Then for* $x > 0$*,*

$$\frac{d(x^a)}{dx} = ax^{a-1}$$

Proof. Denote $y = x^a$. Further we know we can write $x = e^{\ln x}$, so that we have $x^a = e^{a \ln x}$. Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(e^u)}{du}\frac{d(a\ln x)}{dx} \\ &= e^{a\ln x}\frac{a}{x} \\ &= \frac{ax^a}{x} = ax^{a-1} \end{aligned}$$

■

Now, we are ready to consider some more complicated derivatives.

Example 6 *Using the differentiation techniques discussed for logarithmic functions, find* $\frac{d(x^x)}{dx}$.

Solution 6 *Denote* $y = x^x$*. Next, observe that,*

$$\begin{aligned} \ln y &= x \ln x \\ \Rightarrow \frac{d\ln y}{dx} &= x\frac{d\ln x}{dx} + \ln x \\ \Rightarrow \frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\ \Rightarrow \frac{dy}{dx} &= y(1 + \ln x) \\ &= x^x(1 + \ln x) \end{aligned}$$

5 L'Hôpital's Rule

You would have realized how difficult it can get when determining the limit of a quotient of two functions. l'Hôpital's rule is used when determing such a limit when both the numerator and denominator function approaches 0 on the limit. That is,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{0}{0}$$

so that the limit is of an indeterminate form. We will now provide a way of determining a solution to this common problem known as the **zero–over–zero case** of l'Hôpital's rule.

Theorem 10 ***L'Hôpital's rule (zero–over–zero case).*** *Let a be a constant, and let f and g be differentiable functions over (a, b). Further, assume $g(x) \neq 0 \ \forall x \in (a, b)$. Then if $\lim_{x^{a+}} f(x) = 0$ and $\lim_{x \to a^+} g(x) = 0$, and*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Proof. To prove the theorem, we need to first proof the *Generalized Mean-Value Theorem.*

Theorem 11 ***Generalized Mean-Value Theorem.*** *For continuous functions f and g on $[a, b]$, and differentiable on (a, b), assume $g'(x) \neq 0 \ \forall x \in (a, b)$. Then there is constant $c \in (a, b)$ such that,*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof. First define,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

Then observe that,

$$\begin{aligned} h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(b) - g(a)] \\ &= [f(b) - f(a)] - [f(b) - f(a)] = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} h(a) &= f(a) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(a) - g(a)] \\ &= -\frac{f(b) - f(a)}{g(b) - g(a)} \times 0 = 0 \end{aligned}$$

Then by Rolle's Theorem, since $h(a) = h(b) = 0$, there must be a c such that $c \in (a, b)$ such that $h'(c) = 0$. Observe then that,

$$\begin{aligned} h'(c) &= f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0 \\ \Rightarrow \frac{f'(c)}{g'(c)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

which completes the proof. ■

Thus given the generalized mean-value theorem, define $f(a) = g(a) = 0$, implying that f and g are continuous at a. This means that,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Thus by the generalized mean value theorem,

$$\begin{aligned} \lim_{x \to x^+} \frac{f(x)}{g(x)} &= \lim_{x \to x^+} \frac{f(x) - f(a)}{g(x) - g(a)} \\ &= \lim_{x \to x^+} \frac{f'(c)}{g'(c)} = \frac{f'(c)}{g'(c)} \end{aligned}$$

and the proof is complete by denoting $L = \frac{f'(c)}{g'(c)}$. ■

Note that the idea behind the theorem remains true for the cases of $x \to a^-$, $x \to a$, $x \to \infty$, and $x \to -\infty$. To understand how the rule might be used,

Example 7 *Find* $\lim_{x \to 1^+} \frac{x^5 - 1}{x^3 - 1}$.

Solution 7 *Observe first that* $\lim_{x \to 1^+} (x^5 - 1) = 0$ *and* $\lim_{x \to 1^+} (x^3 - 1) = 0$. *By l'Hôpital's rule,*

$$\lim_{x \to 1^+} \frac{x^5 - 1}{x^3 - 1} = \lim_{x \to 1^+} \frac{\frac{d}{dx}(x^5 - 1)}{\frac{d}{dx}(x^3 - 1)}$$

if the righthand side limit of the equality exists. Therefore,

$$\begin{aligned} \lim_{x \to 1^+} \frac{\frac{d}{dx}(x^5 - 1)}{\frac{d}{dx}(x^3 - 1)} &= \lim_{x \to 1^+} \frac{5x^4}{3x^2} \\ &= \lim_{x \to 1^+} \frac{5}{3} x^2 = \frac{5}{3} \end{aligned}$$

It should be noted that sometimes, even after one operation of the l'Hôpital's rule, the issue of $\frac{0}{0}$ remains, so that you would need to apply it again. In other words, you may apply l'Hôpital's rule several times.

Are there other instances besides the $\frac{0}{0}$ case. That would be the **infinity–over–infinity ($\frac{\infty}{\infty}$) case**.

Theorem 12 ***L'Hôpital's rule (infinity–over–infinity) case.*** *For continuous functions f and g defined and differentiable $\forall x > a$ where a is some constant fixed number. Then if $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

This remains true for $x \to a$, $x \to a^-$, $x \to a^+$, or $x \to -\infty$. Further, $\lim_{x\to\infty} f(x)$, $\lim_{x\to\infty} g(x)$, could be both $-\infty$, or they could alternative in sign.

Example 8 *Find* $\lim_{x\to\infty} \frac{x}{e^x}$.

Solution 8

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{e^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} \\ &= \lim_{x\to\infty} \frac{1}{e^x} = 0\end{aligned}$$

Example 9 *Find* $\lim_{x\to\infty} \frac{x^3}{2^x}$.

Solution 9

$$\begin{aligned}\lim_{x\to\infty} \frac{x^3}{2x} &= \lim_{x\to\infty} \frac{3x^2}{2^x \ln 2} \\ &= \lim_{x\to\infty} \frac{6x}{2^x (\ln 2)^2} \\ &= \lim_{x\to\infty} \frac{6}{2^x (\ln 2)^3} = 0\end{aligned}$$

So that the growth rate of the denominator, 2^x is faster than the numerator, which is as expected.

You must keep in mind that l'Hôpital's rule works only if the limit for $\frac{f'}{g'}$ exists. It says nothing about otherwise.

Just because a function is not in quotient form, does not immediately prevent us from applying l'Hôpital's rule. The following examples shows how they could be achieved.

Example 10 *Find* $\lim_{x\to 0^+} x\ln x$.

Solution 10

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= -\lim_{x\to 0^+} x = 0\end{aligned}$$

Example 11 *Find* $\lim_{x\to 0^+} x^x$.

Solution 11

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{x\ln x} \\ &= \lim_{x\to 0^+} e^{\frac{\ln x}{\frac{1}{x}}} \\ &= \lim_{x\to 0^+} e^{x} = 1\end{aligned}$$

The following table helps to highlight the transformations you could perform that permits the use of l'Hôpital's rule.

Form	Name	Method
$f(x)g(x)$; $f(x)\to 0$, $g(x)\to\infty$	zero–times–infinity	Express as $\frac{g(x)}{\frac{1}{f(x)}}$
$f(x)^{g(x)}$; $f(x)\to 1$, $g(x)\to\infty$	one–to–infinity	Find the limit of
$f(x)^{g(x)}$; $f(x)\to 0$, $g(x)\to 0$	zero–to–zero	$\exp(g(x)\ln f(x)) = \exp\left(\frac{\ln f(x)}{\frac{1}{g(x)}}\right)$.