Quantitative Methods in Economics
ECON 271:10
Chapter 6
More on Antiderivatives; Constrained Optimization

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Before we begin, it must be noted that the computation of antiderivatives, although related to derivatives, do not always present itself with the same amount of ease, nor be solvable in the first place. As an example, the function $y = e^{x^2}$ does not have an elementary antiderivative. If you recognize the similarity of the function to that of the *Gaussian Probability Density Function (pdf)*, you would realize then that that means the calculation of the *cumulative distribution function* for Gaussian pdf is not possible!

Further, just because the integrand looks similar, does not mean that the solutions will be the same. Consider the two examples below:

$$\begin{aligned}\int \frac{1}{x^2+1}dx &= \tan^{-1}x + C \\ \int \frac{x}{x^2+1}dx &= \frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

You may obtain or find common tables of antiderivatives (tables of integrals). But due to their brevity, some skills are still required to simplify problems so that the tables can be used effectively. You might also think that computers could be used to solve these problems numerically. However, even there, we meet with limitations, example: how many integrals do wish to calculate; what is domain of the integrand; how many variables are in the integrand which you need to compute the integrals over. We will now discuss some useful skills.

1 Substitution Technique

This technique requires the practitioner to understand that every formula for a derivative provides a corresponding one for an antiderivative/integral. Here some for you to consider:

$$\begin{aligned}
\int x^a dx = \frac{x^{a+1}}{a+1} + C \qquad & \int \frac{1}{x} dx = \ln|x| + C \\
\int e^x dx = e^x + C \qquad & \int \sin x dx = -\cos x + C \\
\int \cos x dx = \sin x + C \qquad & \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \\
\int \frac{1}{1+x^2} dx = \tan^{-1} x + C \qquad & \int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C
\end{aligned}$$

In addition, keep in mind that,

$$\int \frac{f'}{f} dx = \ln|f| + C$$

not to mention all the properties we had discovered previously. Let's consider some examples to refresh our memory.

Example 1 *Compute* $\int \frac{4x^3}{x^4+1} dx$.

Solution 1 *Observe first the the numerator is the derivative of the denominator, so that,*

$$\begin{aligned}
\int \frac{4x^3}{x^4+1} dx &= \ln|x^4+1| + C \\
&= \ln(x^4+1) + C \qquad \text{since } x^4 \text{ is always positive.}
\end{aligned}$$

Example 2 *Compute* $\int \frac{1}{x^3} dx$.

Solution 2

$$\begin{aligned}
\int \frac{1}{x^3} dx &= \int x^{-3} dx \\
&= \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2x^2} + C
\end{aligned}$$

With this in mind, we can now consider the most common technique of integration, that of the substitution method.

Theorem 1 ***Substitution Method.*** *Let $g(u)$ be a continuous function, and further let $h(x)$ be a differentiable function. Denote $G(u)$ as an antiderivative of $g(u)$. Then $G(h(x))$ is an antiderivative of $g(h(x))h'(x)$.*

Proof. Denote $y = G(h(x))$, so that $u = h(x)$. By the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= G'(h(x))h'(x) \\ &= g(h(x))h'(x) \end{aligned}$$

■

Also,

Theorem 2 ***Substitution in the case of a definite integral.*** *Suppose f is a continuous function on $[a,b]$, let $u = h(x)$ be a differentiable function on $[a,b]$ as well, and let g be a continuous function such that,*

$$\begin{aligned} f(x)dx &= g(u)du \\ \Rightarrow f(x) &= g(h(x))h'(x) \end{aligned}$$

Then,

$$\int\limits_a^b f(x)dx = \int\limits_{h(a)}^{h(b)} g(u)du$$

Proof. Denote $G(u)$ as an antiderivative of $g(u)$. By the previous theorem, $G(h(x))$ is an antiderivative of $f(x)$. Therefore,

$$\begin{aligned} \int\limits_a^b f(x)dx &= G(h(x))|_a^b \\ &= G(h(b)) - G(h(a)) \\ &= \int\limits_{h(a)}^{h(b)} g(u)du \end{aligned}$$

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This method is worth attempting under two circumstances:

1. For an integrand $f(x)$, it can be rewritten in the form of the product of a function of $u(x)$, and the derivative of $u(x)$.

2. The integrand becomes a simpler expression when a part of it is written in terms of $u(x)$.

This technique sometimes called the *change of variables* technique. As may be seen in the theorems, the techniques works for both definite and indefinite integrals. You however need to keep in mind that when applying it to the definite integral case, to be sure to replace the interval $[a, b]$, by $[u(a), u(b)]$. The technique's usefulness, and ease will be seen in the following examples.

Example 3 *Compute* $\int 5e^{x^5}x^4dx$.

Solution 3 *Observe first that* $5x^4$ *is the derivative of* x^5*. So let* $u = x^5$*, then* $du = 5x^4dx$*, so that,*

$$\begin{aligned} \int 5e^{x^5}x^4dx &= \int e^u du \\ &= e^u + C \\ &= e^{x^5} + C \end{aligned}$$

Example 4 *Compute* $\int (1 + x^3)^5 x^2 dx$.

Solution 4 *Observe that* $3x^2$ *is the derivative of* $(1 + x^3)$*, which differs from* x^2 *merely by a constant* 3*. So let* $u = (1 + x^3)$*, so that* $du = 3x^2$ *which means that* $\frac{1}{3}du = x^2$*. Therefore,*

$$\begin{aligned} \int (1 + x^3)^5 x^2 dx &= \int \frac{u^5}{3} du \\ &= \frac{u^6}{18} + C \\ &= \frac{(1 + x^3)^6}{18} + C \end{aligned}$$

Example 5 *Compute* $\int\limits_2^4 \frac{x^2+1}{(2x-3)^2}dx$.

Solution 5 *In this case, since the denominator seems to be complicating the integral, let* $u = 2x - 3$*. This means that* $du = 2dx$ *and* $\frac{du}{2} = dx$*. Further, observe that* $x = \frac{u+3}{2}$*.*

Therefore,

$$\begin{aligned}
\int \frac{x^2+1}{(2x-3)^2}dx &= \int \frac{\left[\frac{u+3}{2}\right]^2+1}{2u^2}du \\
&= \int \frac{u^2+6u+13}{8u^2}du \\
&= \int \left(\frac{1}{8}+\frac{3}{4u}+\frac{13}{8u^2}\right)du \\
&= \frac{u}{8}+\frac{3\ln|u|}{4}-\frac{13}{8u}+C \\
&= \frac{2x-3}{8}+\frac{3\ln|2x-3|}{4}-\frac{13}{8(2x-3)}+C
\end{aligned}$$

Therefore the definite integral is,

$$\begin{aligned}
\int_2^4 \frac{x^2+1}{(2x-3)^2}dx &= \left(\frac{2x-3}{8}+\frac{3\ln|2x-3|}{4}-\frac{13}{8(2x-3)}\right)\Bigg|_2^4 \\
&= \frac{9}{5}+\frac{3}{4}\ln 5
\end{aligned}$$

Example 6 *Find the formula for* $\int \frac{dx}{(ax+b)^n}dx$*, for* $n \neq 1$*.*

Solution 6 *Let* $u = ax+b$*, so that* $du = adx$ *. Therefore,*

$$\begin{aligned}
\int \frac{dx}{(ax+b)^n}dx &= \int \frac{1}{au^n}du \\
&= \frac{1}{a}\int u^{-n}du \\
&= \frac{1}{a}\frac{u^{-n+1}}{-n+1}+C \\
&= \frac{(ax+b)^{-n+1}}{a(-n+1)}+C \\
&= \frac{1}{a(-n+1)(ax+b)^{n-1}}+C
\end{aligned}$$

2 Integration by Parts

The substitution method, as you might have surmised, is our reverse engineering the chain rule. We will here discuss *integration by parts* which is obtained from the formula for derivatives for products.

Theorem 3 ***Integration by Parts.*** *If u and v are differentiable functions, and $\int vu'dx$ is an antiderivative of vu', then*

$$uv - \int vu'dx$$

is an antiderivative of uv'. So that,

$$\begin{aligned} \int uv'dx &= uv - \int vu'dx \\ \Rightarrow \int udv &= uv - \int vdu \end{aligned}$$

Proof. Differentiate $uv - \int vu'dx$,

$$\begin{aligned} \frac{d}{dx}\left(uv - \int vu'dx\right) &= \frac{d}{dx}uv - \frac{d}{dx}\left(\int vu'dx\right) \\ &= (uv' + vu') - vu' = uv' \end{aligned}$$

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The essential trick to the application of integration by parts is the labelling of u and dv.

1. v needs to be easily found, or be not "excessively cumbersome".
2. du should not "aggravate" the expression of u.
3. $\int vdu$ should be easier than the original $\int udv$.

We will use the next few examples to familiarize ourselves with the technique.

Example 7 *Compute $\int xe^x dx$.*

Solution 7 *Let $u = x$ so that $du = dx$, and let $dv = e^x dx$ so that $v = e^x$. Observe that for the defining of u, du has simplified the original expression, and we had picked e^x for dv knowing that the derivative and antiderivative of e^x is itself. Now, applying the formula,*

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \end{aligned}$$

Example 8 *Compute $\int x \ln x dx$.*

Solution 8 *Let* $u = \ln x$ *so that* $du = \frac{1}{x}dx$. *and let* $dv = xdx$ *so that* $v = \frac{x^2}{2}$. *Thus,*

$$\begin{aligned} \int x \ln x dx &= \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C \end{aligned}$$

Example 9 *Compute* $\int x^2 e^x dx$.

Solution 9 *Let* $u = x^2$ *so that* $du = 2xdx$, *and let* $dv = e^x dx$ *so that* $v = e^x$. *Thus,*

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2\left(x e^x - e^x\right) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \end{aligned}$$

where the second equality follows from another application of the technique, which we had found two examples prior.

There are other issues to be kept in mind, but because this is a class primarily for applications to Economics, we have not touched on trigonometrical applications, where the other key points and techniques lie. Nonetheless, should you be interested to know more, they are available in most good calculus references.

3 Constrained Optimization

In much of economics, we often find ourselves attempting to optimize some objective of an economic agent, with an understanding that those choices are made subject to often at least one constraint. In consequence such problems are know as constrained optimization problems. Such problems can be described as such,

$$\begin{aligned} &\max_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n) \\ \text{subject to:} \quad & g_1(x_1, \ldots, x_n) \le b_1, \ldots, g_k(x_1, \ldots, x_n) \le b_k \quad , \\ & h_1(x_1, \ldots, x_n) = c_1, \ldots, h_m(x_1, \ldots, x_n) = c_m \end{aligned}$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $g(.)$ represents inequality constraints, while $h(.)$ represents equality constraints. The problem may be reframed as a minimization problem as well,

$$\begin{aligned} &\min_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n) \\ \text{subject to:} \quad & g_1(x_1, \ldots, x_n) \ge b_1, \ldots, g_k(x_1, \ldots, x_n) \ge b_k \quad , \\ & h_1(x_1, \ldots, x_n) = c_1, \ldots, h_m(x_1, \ldots, x_n) = c_m \end{aligned}$$

realizing you have to pay attention to the set for the inequality constraints $g(.)$. We will deal with only equality constrained problems at this juncture.

Consider first a simplest two variable equality constrained problem,

$$\begin{aligned} & \max_{x_1,x_2} u(x_1, x_2) \\ \text{subject to:} \quad & m = h(x_1, x_2) \end{aligned}$$

where in economics, $u(x_1, x_2)$ would be an agent's utility function, subject to the some constraint (most commonly a pecuniary constraint called a budget constraint). x_i's represents the quantities of good i an individual could chose, and m is the monetary income. You will commonly observe prices as a parameter within the model, but since we are keeping our discussion general at the moment, we will ignore it for now. What this problem requires of us is to find the highest tangency between the level set generated by $u(x_1, x_2)$ and the budget constraint $h(x_1, x_2)$. On the x_1 and x_2 plane, where $x_i \in \mathbb{R}^+ + \{0\}$, the gradient of the level set would be $\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$. On the other hand, that for the constraint would be $\frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}}$, so that at the optimum, we have,

$$\begin{aligned} \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} &= \frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}} \\ \Rightarrow \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial h}{\partial x_1}} &= \frac{\frac{\partial u}{\partial x_2}}{\frac{\partial h}{\partial x_2}} = \lambda \end{aligned}$$

This thus imply that,

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= \lambda \frac{\partial h}{\partial x_1} \\ \Rightarrow \frac{\partial u}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} &= 0 \\ \& \quad \frac{\partial u}{\partial x_2} &= \lambda \frac{\partial h}{\partial x_2} \\ \Rightarrow \frac{\partial u}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} &= 0 \end{aligned}$$

However, the above two equations cannot identify all the unknowns which are (x_1, x_2, λ). The third equation is obtained from the constraint itself, $h(x_1, x_x) - c = 0$.

These conditions can be quickly obtained by setting up the *Lagrangian Function*,

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(h(x_1, x_2) - c)$$

Then the three previous equations can be obtained quickly by finding, $\frac{\partial \mathcal{L}}{\partial x_1}$, $\frac{\partial \mathcal{L}}{\partial x_2}$, $\frac{\partial \mathcal{L}}{\partial \lambda}$, and setting the derivatives equal to 0. The λ is typically refered to as the Lagrange multiplier, and its value tells us the importance the constraint has on the problem. The key observation that could be made is that we have changed a constrained problem, into an unconstrained one when we use a Lagrangian function. Though in consequence, we gain an additional variable, λ that needs to be solved. Another observation that needs to be made in that we need $\frac{\partial h}{\partial x_i} \neq 0$, as is clear from the above. This is known as a constraint qualification. We can state the above discussion formally now.

Theorem 4 *Let u and h be continuous and differentiable functions, for $(x_1, x_2) \in \mathbb{R}^2$. Let (x_1^*, x_2^*) be a solution to the problem,*

$$\begin{aligned} &\max_{x_1, x_2} u(x_1, x_2) \\ \text{subject to:} \quad &h(x_1, x_2) = c \end{aligned}$$

Suppose (x_1^*, x_2^*) is not a critical point of h. Then, there is a real number λ^*, such that $(x_1^*, x_2^*, \lambda^*)$ is a critical point of the Lagrangian function,

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda[h(x_1, x_2) - c]$$

such that,

$$\frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_1} = 0 \quad \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial x_2} = 0 \quad \frac{\partial \mathcal{L}(x_1^*, x_2^*, \lambda^*)}{\partial \lambda} = 0$$

The theorem holds for minimization as well. To ascertain if we are truly maximizing or minimizing the objective, we would need to verify the semi-definiteness of the Lagrangian, which we will leave to a more advanced class. Essentially, what we would need to ascertain is whether the unconstrained problem represented by the Lagrangian function, is "concave" or "convex". To consolidate the idea, we will try several examples.

Example 10 *Solve the following constrained optimization problem,*

$$\begin{aligned} &\max_{x_1, x_2} f(x_1, x_2) = x_1 x_2 \\ \text{subject to:} \quad &h(x_1, x_2) \equiv x_1 + 4x_2 = 16 \end{aligned}$$

Solution 10 *Setting up the lagrangian,*

$$\mathcal{L}(x_1, x_2) = x_1 x_2 - \lambda(x_1 + 4x_2 - 16)$$

The set of partial derivatives conditions are thus,

$$\begin{aligned}\frac{\partial\mathcal{L}(x_1,x_2)}{\partial x_1} &= x_2-\lambda=0\\ \frac{\partial\mathcal{L}(x_1,x_2)}{\partial x_2} &= x_1-4\lambda=0\\ \frac{\partial\mathcal{L}(x_1,x_2)}{\partial \lambda} &= x_1+4x_2-16=0\end{aligned}$$

Reexpressing the first two conditions in terms of the Lagrangian multiplier, and equating them to each other yields,

$$\begin{aligned}x_2 &= \frac{x_1}{4}\\ \Rightarrow 2x_1 &= 16 \qquad \text{Substituting the equality into the third condition.}\\ \Rightarrow x_1 &= 8\\ \therefore x_2 &= 2\\ \&\lambda &= 2\end{aligned}$$

and we can state the solution succinctly as $(8,2,2)$*, for* (x_1,x_2,λ)*.*

Example 11 *Compute the following,*

$$\begin{aligned}&\max_{x_1,x_2} f(x_1,x_2)=x_1^2x_2\\ \text{subject to:}\qquad &2x_1^2+x_2^2=3\end{aligned}$$

Solution 11 *Setting up the Lagrangian function, we obtain,*

$$\mathcal{L}(x_1,x_2)=x_1^2x_2-\lambda\left(2x_1^2+x_2^2-3\right) \tag{1}$$

This gives the following conditions,

$$\begin{aligned}\frac{\partial\mathcal{L}(x_1,x_2)}{\partial x_1} &= 2x_1x_2-4\lambda x_1=2x_1(x_2-2\lambda)=0\\ \frac{\partial\mathcal{L}(x_1,x_2)}{\partial x_2} &= x_1^2-2\lambda x_2=0\\ \frac{\partial\mathcal{L}(x_1,x_2)}{\partial \lambda} &= 2x_1^2+x_2^2-3=0\end{aligned}$$

From the first condition, we see that there are two cases we need to examine, the first is when $x_1=0$*, and the second is when* $x_2=2\lambda$*.*

Considering the first case when $x_1 = 0$. Substituting this into the third condition, in other words, the constraint, we obtain $x_2 = \pm\sqrt{3}$. The second condition tells us then that $\lambda = 0$. This from this first case, we have two solutions, $(0, \sqrt{3}, 0)$ and $(0, -\sqrt{3}, 0)$, where the triplet of elements/coordinates are arranged as (x_1, x_2, λ) as before.

Considering the second case, when $x_2 = 2\lambda$ which implies that $\lambda = \frac{x_2}{2}$, when $x_1 \neq 0$. Substituting this into the second condition yields $x_1^2 - x_2^2 = 0$ implying that $x_1^2 = x_2^2$. Substituting this latter condition into the third condition/constraint gives us $3x_1^2 = 3$ or $x_1 = \pm 1$ and in consequence $x_2 = \pm 1$. This thus gives us four different permutations for the solution: $(1, 1, 0.5)$; $(1, -1, -0.5)$; $(-1, 1, 0.5)$; $(-1, -1, -0.5)$. To verify which of the solutions yield the maximum, we can substitute them into the objective function,

$$\begin{aligned} f(1,1) &= f(-1,1) = && 1 \\ f(1,-1) &= f(-1,-1) = && -1 \\ f(0,\sqrt{3}) &= f(0,-\sqrt{3}) = && 0 \end{aligned}$$

so that the solutions are $(1, 1, 0.5)$, and $(-1, 1, 0.5)$. Observe as well that $(1, -1, -0.5)$ and $(-1, -1, -0.5)$ minimizes the constrained problem instead.

What if we have more than one constraint, and more than two variables? The problem just becomes more complicated, but the method holds, and the following theorem highlights how we may use the method, and set up the Lagrangian.

Theorem 5 *Let u and h be continuous and differentiable functions, for $\mathbf{x}' = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$. For the maximization problem*

$$\begin{aligned} & \max_{x_1, x_2, \ldots, x_n} u(x_1, x_2, \ldots, x_n) \\ \text{subject to:} \quad & h_1(x_1, x_2, \ldots, x_n) = c_1 \\ & \vdots \\ & h_m(x_1, x_2, \ldots, x_n) = c_m \end{aligned}$$

suppose $(x_1^*, x_2^*, \ldots, x_n^*)$ satisfies all the m constraints (we say it is in the constraint set), and that it is a (local) maximum of $u(.)$ as well. In addition, suppose it satisfies the non-degenerate constraint qualification at $\mathbf{x}^{*\prime} = [x_1^*, x_2^*, \ldots, x_n^*]$, in other words, the Jacobian *matrix of first derivatives cannot be a matrix of zeros.*

$$Dh(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \frac{\partial h_2}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_2}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{bmatrix} \neq \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

Then there exists $\mathbf{\Lambda}^* = (\lambda_1^*, \dots, \lambda_m^*)$ such that $(\mathbf{x}^*, \mathbf{\Lambda}^*)$ is a critical point of the Lagrangian function,

$$\mathcal{L}(\mathbf{x}, \mathbf{\Lambda}) = u(\mathbf{x}) - \lambda_1(h_1(\mathbf{x} - c_1)) - \dots - \lambda_m(h_m(\mathbf{x} - c_m))$$

so that,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1}(\mathbf{x}, \mathbf{\Lambda}) = 0 \quad &, \dots, \quad \frac{\partial \mathcal{L}}{\partial x_n}(\mathbf{x}, \mathbf{\Lambda}) = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1}(\mathbf{x}, \mathbf{\Lambda}) = 0 \quad &, \dots, \quad \frac{\partial \mathcal{L}}{\partial \lambda_m}(\mathbf{x}, \mathbf{\Lambda}) = 0 \end{aligned}$$

To see how it works, let us consider an example.

Example 12 *Compute the following,*

$$\begin{aligned} \max_{x,y,z} f(x,y,z) &= xyz \\ \text{subject to:} \quad h_1(x,y,z) &= x^2 + y^2 = 1 \\ \& \quad h_2(x,y,z) &= x + z = 1 \end{aligned}$$

Solution 12 *First to see if the non-degenerate constraint qualification is satisfied,*

$$\begin{bmatrix} \frac{\partial h_1}{\partial x} = 2x & \frac{\partial h_1}{\partial y} = 2y & \frac{\partial h_1}{\partial z} = 0 \\ \frac{\partial h_2}{\partial x} = 1 & \frac{\partial h_2}{\partial y} = 0 & \frac{\partial h_2}{\partial z} = 1 \end{bmatrix}$$

and we see that the first row of the matrix is all zero if and only if $x = y = 0$, which if true would violate the first constraint, so that there is nothing to worry about. Technically, we say the Jacobian matrix is full rank, and has a rank of 2 (Should you be interested to find out more about matrices, you should take the another class in Math for Economics, or courses in Matrix and Linear Algebra.). Setting up the Lagrangian, we have

$$\mathcal{L}(x, y, z, \lambda_1, \lambda_2) = xyz - \lambda_1(x^2 + y^2 - 1) - \lambda_2(x + z - 1)$$

The set of conditions that needs to be met now are,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x}(x, y, z, \lambda_1, \lambda_2) &= yz - 2\lambda_1 x - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial y}(x, y, z, \lambda_1, \lambda_2) &= xz - 2\lambda_1 y = 0 \\
\frac{\partial \mathcal{L}}{\partial z}(x, y, z, \lambda_1, \lambda_2) &= xy - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1}(x, y, z, \lambda_1, \lambda_2) &= x^2 + y^2 - 1 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2}(x, y, z, \lambda_1, \lambda_2) &= x + z - 1 = 0
\end{aligned}$$

Since the second and third conditions define the Lagrange multiplier uniquely, we can express them in simpler form,

$$\begin{aligned}
\lambda_1 &= \frac{xz}{2y} \\
\lambda_2 &= xy
\end{aligned}$$

So that substituting these values into the first condition gives us,

$$\begin{aligned}
yz - \frac{x^2 z}{y} - xy &= 0 \\
\Rightarrow y^2 z - x^2 z - xy^2 &= 0
\end{aligned}$$

From the fourth and fifth conditions,

$$\begin{aligned}
y^2 &= 1 - x^2 \\
\&z &= 1 - x
\end{aligned}$$

Substituting these into the prior equation thus yields,

$$\begin{aligned}
(1 - x^2)(1 - x) - x^2(1 - x) - x(1 - x^2) &= 0 \\
\Rightarrow 1 - x - x^2 + x^3 - x^2 + x^3 - x + x^3 &= 0 \\
\Rightarrow 1 - 2x - 2x^2 + 3x^3 &= 0 \\
\Rightarrow (x - 1)(3x^2 + x - 1) &= 0
\end{aligned}$$

The quadratic portion has the roots, $\frac{1}{6}(-1 \pm \sqrt{13})$*, which are approximately* -0.7676 *and* 0.4343*, and the third root as is clear is 1 (As is easy to verify,* $x = 1$ *yields an undefined* λ_1*, so cannot be a solution, and further yield* $f = 0$*, which is neither a maximum nor* *a minimum). The four possible solutions are thus:* $(0.4343, 0.9008, 0.5657, 0.1364, 0.3912)$*;* $(0.4343, -0.9008, 0.5657, -0.1364, -0.3912)$ *;* $(-0.7676, 0.6409, 1.7676, -0.3205, -0.4920)$*;* $(-0.7676, -0.6409, 1.7676, 0.3205, 0.4920)$*. You may then verify that the maximizer is at* $(-0.7676, -0.6409, 1.7676, 0.3205, 0.4920)$*.*

This way of altering a constrained problem into an unconstrained one works also for inequality constraints, which is discussed in other more advanced classes.