Mathematical Economics (ECON 471)
Lecture 1
Revision of Calculus

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Although mathematics has been in use in Economics for close to a century now, it is over the past five decades that it has been the "language" of choice in Economics. The principal rationale for our principal reliance on it is due to the fact that as our mind tries to grapple with more variables of concern in understanding the economy, our tendency towards laziness in reasoning becomes more apparent, and liable to errors.

We will first collate the mathematical concepts we have been using through your academic career thus far, and build upon them, beginning with this simple revision on Calculus.

1 One Variable Calculus

1.1 One Variable Functions & their Derivatives

A **function**, as you should recall, is a rule which maps/assigns a number on the number line, extending from $-\infty$ to ∞ (simply $(-\infty, \infty)$, or what commonly denote as $\mathbf{R}^1$), to each number on $\mathbf{R}^1$. It is common to denote this function in a form $f(.)$, where the alphabet denoting the function is totally arbitrary. The input number from $\mathbf{R}^1$ into the function is commonly denoted by a alphabet as well, such as x, and another for the output variable, say y. The former is called the **independent variable** (or in economics, it is commonly referred to as the **exogenous variable**), while the latter is referred to as the **dependent variable** (or rather without imagination, the **endogenous variable** in economics). The simplest example of a function would be,

$$y = f(x) = x$$

The simplest functions we can deal with are typically monomials, which are of the form $f(x) = ax^k$ where both a and k are just positive integers on $\mathbf{R}^1$. A simple example being

$$y = f(x) = 2x^2$$

where $a = k = 2$. a is referred to as the **coefficient**, and k is the **degree**. If you were to add/subtract a series of monomials, you would get a **polynomial**, such as

$$y = f(x) = 7x^{12} + 10x^6 - 2x^3$$

In a polynomial, the highest degree in the function is call the degree of the polynomial, which in the above case is 12.

There are of course other functional forms which may become more common as you advance higher in your academic career in economics. Ratios of polynomials are known as **rational functions**, such as,

$$y = f(x) = \frac{x^2 - 2}{x^3 + 3}$$

There is also the **exponential functions** where the independent variable is the exponent, such as

$$y = f(x) = 10^x$$

and the **trigonometric functions**, such as

$$y = f(x) = \sin(x)$$

Visually, most functions can be illustrated as a **graph** (in the single variable case, the graph contains all the relationships $(x, y) = (x, f(x))$ on the **Cartesian Plane**), and we typically describe it as an **increasing function** (if the graph tends upwards moving from left to right), or **decreasing function** (if the graph tends downwards moving from left to right). It is also perfectly possible to have the graph with a turning point, where the combination may be first increasing, then decreasing, or vice versa. In fact, it is these sorts of functions that we most commonly deal with. When a function first increases then decreases around x_0, the turning point is then $(x_0, f(x_0))$, and the graph of the function lies below $(x_0, f(x_0))$. We say the turning point is a **local** or **relative maximum**. If all other points on the support of the graph lies entirely below the turning point, we call the turning point a **global** or **absolute maximum**. An example being the function,

$$y = f(x) = -x^2$$

If the vice versa is true, in that the function is first decreasing, then increasing around $(x_0, f(x_0))$. We would call the turning point a **local** or **relative maximum**. Again if the entire graph lies above it, it is a **global** or **absolute minimum**. An example of such a function is,

$$y = f(x) = x^4$$

The idea of local versus global maximums and minimums are particularly prevalent in economics since it is common to have the independent variable defined on a subset of $\mathbf{R}^1$. In complicated functions where we are unsure if the turning point is a global or local turning point, we say it is a local minimum or maximum within the subset. The set of x's which $f(x)$ is defined is known as the **domain** of $f(.)$. A natural example is the function $y = \frac{1}{x}$ where it is defined on $\mathbf{R}^1 - \{0\}$, in other words all points on the real line excluding the origin. A natural example in economics is when we talk about the production choice of a firm. When the firm makes a choice regarding its optimal input combinations, it is dependent on the cost function the firm faces, and since price of inputs are at the cheapest, free, the domain of the cost are all positive integers including zero. Here, letting the number of output be x, we can describe the domain as $\mathbf{R}_+ + \{0\}$, in other words all points on the positive half of the real line inclusive of zero.

As a further extension on notation, letting the domain of x be D, and the function maps x onto R, where both $D, R \subset \mathbf{R}^1$, we can write the entire idea as,

$$f : D \longrightarrow R$$

Neither the domain or range of x need to be completely closed, in that the left and right most real number may be either included or excluded as a couple, or independently. Let the domain of x be such that it is bounded on the left and right (we can say left and right here because we are talking about $\mathbf{R}^1$) by $\underline{x}$ and $\overline{x}$. If the set of values of the domain includes the boundary, it means the range is

$$[\underline{x}, \overline{x}] \equiv \{x \in \mathbf{R}^1 : \underline{x} \leq x \leq \overline{x}\}$$

and we say the domain is a **closed interval**. Similarly, it the set of values of the domain excludes the boundary,

$$(\underline{x}, \overline{x}) \equiv \{x \in \mathbf{R}^1 : \underline{x} < x < \overline{x}\}$$

then we say the domain is a **open interval**. It is also perfectly possible that only one of the extreme points are included or excluded, and we say the interval is **half-open** or **half-closed**. Finally, bringing all this together, we have then also five types of **infinite intervals** for which you can find them in your textbook.

The simplest functions you can deal with would be a polynomial of degree 0, which is just a flat graph, or we typically call it a constant function, such as $f(x) = a$, where a is a number on $\mathbf{R}^1$. The simplest "interesting" function is the polynomial of degree 1, or the **linear function**, which you have dealt with since your first year economics courses, in the form of the demand and supply functions. You would have noticed we had previously written those equations in the form,

$$y = f(x) = mx + b$$

where both m and b are numbers on $\mathbf{R}^1$. The latter is the intercept, while m is the slope of the function. The slope describes to us the steepness/gentleness of the graph of the line. The most natural way to measure this is to examine the change in the y coordinate with a unit change in the x coordinate. That is starting from (x_1, y_1), if we move x_1 say to the right by one unit, we would see a change in the y coordinate by Δy. The ratio of this relative change is known as the slope, m. To compute and convince yourself that m is the slope,

$$\begin{aligned}
\frac{(y_1 + \Delta y) - y_1}{1} &= \frac{\Delta y}{1} \\
&= \frac{m(x_1 + 1) + b - (mx_1 + b)}{1} \\
&= m
\end{aligned}$$

More generally, let two arbitrary points in a two dimensional space be (x_0, y_0) and (x_1, y_1). Then the slope of this linear line segment passing through both points is,

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

Your prior training in calculus would have informed you that the particular points chosen for such linear functions do not change the slope. In economics, this is an important concept since in essence, the slope tells us the **rate of change** of the function in question. The term used for this rate of change is dependent on the function or question on hand. In economics the rate of change concept is most commonly thought of in terms of marginal changes in cost, income, wages, etc.

However, most of the functions we deal with in economics are nonlinear. With nonlinear functions, as is evident from it's graph, the slope is dependent on the location at which you are measuring it. The slope for a nonlinear function, as you should already know is defined as the tangent to the point on the graph of the function. The tangent can be found by finding the **derivative** of the nonlinear function, and calculating the solution as the point of interest. The process of taking a derivative should be familiar to you by now, and we will very briefly cover the basic rules.

Theorem 1 *For a positive integer a and constant k, the derivative of $f(x) = kx^a$ is just $f'(x) = kax^{a-1}$, and the slope at x_0 is just $f'(x_0) = kax_0^{a-1}$.*

where following convention $f'(.)$ is just the first derivative of the function $f(.)$.

Other common nonlinear function that is commonly used in economics include the exponential and logarithmic functions. Recall their derivatives?

Theorem 2 *Both e^x and $\ln(x)$ are continuous functions on their domains, and both are consequently differentiable for every order. Their derivatives are,*

1.
$$\frac{de^x}{dx} = e^x$$

2.
$$\frac{d\ln(x)}{dx} = \frac{1}{x}$$

Further, for a continuous function $h(x)$,

1.
$$\frac{de^{h(x)}}{dx} = h'(x)e^{h(x)}$$

2.
$$\frac{d\ln(h(x))}{dx} = \frac{h'(x)}{h(x)}$$

for $h(x) > 0$.

More generally, an exponential function is $f(x) = b^x$, where b is the base, and is just a positive constant. Its derivative is,

Theorem 3 *For $b > 0$, the derivative of the exponential function b^x is*

$$\frac{db^x}{dx} = \ln(b)b^x$$

To complete the revision of your calculus for a single variable, recall the following rules,

Theorem 4 *For arbitrary constant a and integer k, and functions $f(.)$ and $g(.)$, where both functions are continuous and consequently differentiable, we have the following rules of differentiation pertaining to multiple functions.*

1.
$$\frac{d(f \pm g)(x)}{dx} = (f \pm g)'(x) = f'(x) \pm g'(x)$$

2.
$$\frac{d(af)(x)}{dx} = (af)'(x) = a(f'(x))$$

3. ***Product Rule***:
$$\frac{d(f.g)(x)}{dx} = (f.g)'(x) = f'(x)g(x) + g'(x)f(x)$$

4. ***Quotient Rule***:
$$\frac{d(f/g)(x)}{dx} = \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

5. ***Power Rule***:
$$\frac{df(x)^k}{dx} = kf(x)^{k-1}f'(x)$$

Relationships between variables of concern in economics are often more convoluted in nature. For instance, in health economics, strictly speaking, healthcare services rendered by healthcare professionals is not a "typical" good in the sense that an individual desires it. However, what the agents do desire is their personal health, so that the demand of healthcare products and services is derived through an agent's preference for good as opposed to bad health. In mathematical terms, we have a function in a function, such as

$y = f(g(x))$ which may also be written as $y = (f \circ g)(x)$. As you well know by now, what the derivative of y with respect to x is

$$\frac{d}{dx} f(g(x)) = f'(g(x)).g'(x)$$

which is just the **Chain Rule**.

You have to always keep in mind under what circumstances a function is differentiable. A derivative exists (for the function on hand) if at the point of consideration, the tangent to that point exists. The key point to note is whether the graph of the function is a smooth curve on the support you are concerned with. If it is, then it is differentiable on that support. This then means that although on the entire support of a function, it may have a discontinuity, as long as the segment you are focused on is continuous, its derivative can be found, and is meaningful to you. Further, if the function's derivative exists, the derivative itself is a function. This then leads on to the next question of whether we can likewise find the derivative of the function, and what it means, the topic of higher derivatives which will come shortly below.

Are there even more complicated cases? One common case we sometimes deal with is when we observe a final relationship, but we are more concerned with the inverse of the relationship. Are there rules then that allow us to analyse the inverse of a function directly? There is, and it is due to the **Inverse Function Theorem**. Consider the case of $y = f(x)$. As long as the function $f(.)$ is continuous and differentiable on the support of $x \in \mathbf{R}^1$, then $f(.)$ is invertible, or its inverse exists, so that we can write $f^{-1}(y) = x$. This inverse $f^{-1}(.)$ is then also likewise continuous and differentiable on $y \in \mathbf{R}^1$. What is the derivative of this inverse function with respect to y? In other words, what is $df^{-1}(y)/dy$.

$$\begin{aligned}
f^{-1}(y) &= x \\
\Rightarrow y &= f(f^{-1}(y)) \\
\Rightarrow 1 &= f'(f^{-1}(y)).f^{-1\prime}(y) \\
\Rightarrow \frac{df^{-1}(y)}{dy} = f^{1\prime}(y) &= \frac{1}{f'(f^{-1}(y))}
\end{aligned}$$

What are the implications of being able to find the derivative of a function? What you would have learnt in your earlier mathematics classes is that the derivatives tells you whether as you graph the function, whether it should be increasing or decreasing on the support of the function, in other words, where x lies. However, if the derivative is itself a function of x, your independent variable or right hand side variable, it would mean that the slope of the curve is dependent on the value of x you are looking at, which brings us to the next question. Is the slope of the graph increasing or decreasing?

Besides the first order derivative, it is also very important in economics to know the second order derivative of the function we are dealing with, as it is in mathematics and statistics. Ask yourselves this question, how do you know if the choice made by the firm in profit maximizing, is indeed profit maximizing? What shape/structure should your profit function be if profit is to be maximized? Put another way, if y denotes profit, and x denotes the input choices, then within the support or domain of x, can the graph of the profit function, $y = f(x)$ be concave or convex if the solution is in the interior? If you are thinking that since it is a maximizing problem, we would need a peak, so that the function has to be concave, you would be correct. The manner in which we can verify this is through examining the rate of change of the slope. If the function needs to be initially increasing, which is the case for profit maximization, then we need a slowing down such that the slope decreases as x increases. A quick way to verify this would be to simply check the second order derivative of the function which effectively tells us the "rate of change of the rate of change". In the case of concavity, we need the second derivative to be then negative. However, note importantly that we can perform the second order derivative if and only if it the function is concave, and at least twice differentiable. Of course, you would have learned even higher order derivatives, but for economics, the second derivative will usually suffice.

With the ability the determine the first and second order derivatives, you may then sketch the graph, and know how it behaves on the support of x. This understanding tells you whether you have a concave or convex function, or whether the graph has an inflection point where it switches from concave to convex, or vice versa. This has important implications for economics, since it determines whether you would have a solution to your question on hand. For instance, with the support of x which may be only a subset of $\mathbf{R}^1$, the graph may have a highest point, but if the function has an even higher point outside of the support of x, your solution gives you only a local maximum, or $f(x)$ is just a local maximum, and where this maximum is achieved at x^*, then x^* is the **local maxima**. x^* is the global maxima if and only if $f(x^*)$ is a maximum attainable on the entire support of the function. You can talk about the reverse as well in a similar manner, i.e. global versus local minimum and minima. To consolidate this revision, we will apply these skills to some simple economic questions.

2 Applications

Let us begin with an example from your first year. Consider the elasticity of demand which is defined as the percentage change in the quantity demanded of a good, say x, as a result of a change in its own price, p. What does this mean mathematically. Let $x(p)$ be the quantity demanded of the good. Then the rate of change in the quantity demanded of the good is just the first order derivative $dx(p)/dp = x'(p)$. However, this is measured in term of the units of price and quantity. To alter this formula into percentages which is defined as the elasticity of demand, all we need to do is to multiply $p/x(p)$, since this then gives us

$$\begin{aligned} \epsilon &= \frac{dx(p)}{dp}\frac{p}{x(p)} \\ &= \frac{x'(p).p}{x(p)} \\ &= \frac{\frac{\Delta x(p)}{x(p)}}{\frac{\Delta p}{p}} \\ &= \frac{\text{\% Change in quantiy demanded of good } x}{\text{\% Change in price of good } x} \end{aligned}$$

Of course assuming the demand function $x(p)$ is continuous.

Let's look at another simple example. Why do we depict the average cost (AC) and marginal cost (MC) curve such that the MC intersects the AC from the bottom, at the minimum point of the AC? Can we prove this to be true. Let $C(x)$ be the cost function of the good, and x be the quantity produced by the firm. Then by definition of an average

$$AC(x) = \frac{C(x)}{x}$$

What we need to realize first is that the cost function is increasing, and it is doing so at an increasing rate, in other words, $C'(x) = MC(x)$ is not only increasing, but convex as well. Let us examine how the slope behaves. The slope of the AC is just,

$$\begin{aligned} AC'(x) &= \frac{xC'(x) - C(x)}{x^2} \\ &= \frac{C'(x) - [C(x)/x]}{x} \\ &= \frac{MC(x) - AC(x)}{x} \end{aligned}$$

Since quantity x is a nonnegative number (you cannot "un-produce" something that does not exist) the slope of the AC is dependent on the sign of $MC(x) - AC(x)$, so that it is increasing if $MC(x) > AC(x)$ and decreasing when $MC(x) < AC(x)$. This then implies that the AC curve must be initially decreasing, then increasing. The turning point occurs when $MC(x) = AC(x)$ for some $x = x^*$, which is the minima of $AC(x)$. Whether we can say this minima is a local or global minima depends on the functional form of the cost function.

Let's do something even more substantial. You have learned in Producer Theory that economist typically model firms as profit maximizing entities. The profit is measured as the difference between the revenue $R(x)$ and cost function $C(x)$, where as before x is the quantity choice that a firm makes. In short,

$$\Pi(x) = R(x) - C(x)$$

Technically, the profit function maps $x \in \mathbf{R}^1 + \{0\}$ onto $\mathbf{R}^1$. That is the profits can be either positive or negative. Next ask yourself the following question, how should the functions behave for the firm's choice to be really profit maximizing. That is at the optimal choice of x^*, what kind of assumptions do you need of $R(.)$ and $C(.)$, for the $\Pi(x^*)$ to be a global/local maximum. What you should realize first is that you need the profit function to be increasing first, and in addition to that, it should be concave, failing which you will have only a corner solution (a solution obtained at the upper/lower bound of the support of x). The revenue function is principally derived from the production function, which based on our standard assumptions of diminishing returns, implies that it would be increasing and concave. The cost function as before, should likewise be increasing, but convex (increasing at an increasing rate), due perhaps to the constraints of technology. As a rule, the sum of two concave (convex) functions is a concave (convex) function. So that for the case on hand, those independent assumptions gives you a concave profit function. However, that does not guarantee you would have an interior solution (a solution strictly within the support of x). For that to happen, the initial rate of return on production should be greater than the cost, failing which $x^* = 0$. Alternatively, within the technological possibility in the production, the revenues are increasing faster than cost for the entire support of $x \in [0, \overline{x}]$, then $x^* = \overline{x}$. Given this preliminary comments, we can now derive the optimal choice condition. For the firm to profit maximize, they have to obtain the local maximum at x^*. This occurs, when the slope of the profit function is

0. In other words,

$$\begin{aligned}
\frac{d\Pi(x^*)}{dx} &= \frac{dR(x^*)}{dx} - \frac{dC(^*)}{dx} \\
&= R'(x^*) - C'(x^*) \\
&= 0 \\
\Rightarrow R'(x^*) &= C'(x^*) \\
\Rightarrow MR(x^*) &= MC(x^*)
\end{aligned}$$

which is your standard **Marginal Revenue** equating with **Marginal Cost** condition, and it defines your solution.