Intermediate Microeconomics II, ECON 301
General Equilibrium III: Welfare

We are done with the vital concepts of general equilibrium. Its power principally lies with its ability to consider more than a single player in several markets. An equilibrium is one where there are no additional gains to trade, i.e. that is must be **Pareto Efficient**. But as we have noted, it says nothing about the distribution of welfare within the economy. Although Pareto Efficiency is a desirable goal, it is none unique, in the sense that beginning from any initial endowment point, we can find a different Pareto Efficient Equilibrium point altogether. So the question then which of them is the best for all of us. Who should get a bigger piece of the pie? Should everyone get the same portion of pie?

To address the issue of distribution, we will be examining how we could aggregate the utility or welfare of the constituents of a society, and we consequently term such a function, a **Welfare Function**. What this function would do is to allow us to rank different allocations.

1 Aggregating Preferences

We will now examine an interesting phenomenon about achieving consensus of opinion. Let there be three agents names in table 1. Let's assume that all these agents have transitive preferences (i.e. if $x \succ y$, and $y \succ z$, then $x \succ z$). Let x, y and z be allocations over a full spectrum of goods, which the three agents has personal preference over, and which are ranked below in their respective columns from the most important at the top to the least important. What we want to examine is if it is possible to find some manner in which to "aggregate "individual preferences into a **social preference**. Another way to think about this little experiment is that we are examining the problem of social decision making in its most general form.

- **Majority Voting**: One possibility is to have individual vote for their socially preferred allocation. That is we are seeking agreement socially that $x \succ y$ if

Table 1: Preferences that lead to Intransitive Voting

Tim	Matthew	Chelsea
x	y	z
y	z	x
z	x	y

a majority of individuals agree. The problem with this idea is that it may not generate a transitive social preference ordering. A case in point is the example of table 1. Majority of individuals there agree that $x \succ y$. But so to do they agree that $y \succ z$, and $z \succ x$. If you see the logic of this "aggregation", the final preference ranking is a violation of transitive preferences! And you wonder why there exists conflict! This essentially means that there is no "best "alternative. This also means that the path society takes is dependent on the order of the vote. You should be able to see this.

- **Rank Order Voting**: This method of voting involves all agents assigning numbers that indicates the ranking of the allocation. So the allocation with the smallest number is best, and the largest the worst. However, this method of voting also yields a similar problem where the outcome of the ranking is dependent on the alternatives or the number of alternatives considered at anytime. Examine table 2 for this, where we pit agents Sara and Lindsay.

What the above reveals is that the outcome of a vote can be manipulated by a knowledgeable and cunning agent! Might that be you? That is by choosing the sequence of allocations (or issues) considered, an agent could get his way with the vote in majority voting. While in Rank Order Voting, by introducing alternatives, varying them, an agent could affect ranking that favors his preferences.

You might be wondering if there is a mechanism out there that might allow us to aggregate preferences in a consistent manner. Before we do that, lets note what we would ideally like this mechanism to achieve:

1. The Social Preferences should remain complete, reflexive, and transitive.

Table 2: Choices are dependent on the number of Alternatives Considered

Sara	Lindsay
x	y
y	z
z	x

2. If everyone says $x \succ y$, then the social preference should also exhibit $x \succ y$.

3. The choice between x and y should be dependent only on how agents rank the two, and not how they rank other alternatives.

In relation to the above aims, Kenneth Arrow proved the following:

Arrow's Impossibility Theorem: *If a social decision mechanism satisfies properties 1, 2, and 3, then it must be a dictatorship: all social rankings are the rankings of one individual.*

It says that our aims that we set forth above are inconsistent with Democracy, i.e. there is no perfect way to make social decisions. Put another way, if we wish to achieve some kind of aggregation to form this illusive social preference, we have to give up on one of the above.

2 Social Welfare Functions

If we are willing to eliminate property 3, some types of rank order voting are viable. Let individuals preference be characterized by a utility function over allocations, u_i, such that $x \succ y \Rightarrow u_i(x) > u_i(y)$, where i denotes the agent. Let there be n agents. Then method is to simple sum over everyone's utility for each allocation, and compare the values derived from differing allocations. If one sum is greater than the other the one that yields the greatest utility sum wins out:

$$\sum_{i=1}^{n} u_i(x) > \sum_{i=1}^{n} u_i(x)$$

This method is highly arbitrary in two ways:

1. The utility function over the allocations is not unique.
2. Why should we sum everything together? Why not multiplicative. How about the weighting, since the above weight is implicitly the same for all. Can not we sum some function of the individual utilities, such as taking the squares of the utility, or log?

The first point pertains to the idea that we could always scale up or down the utility measure. The second point relates to the idea that we might want to give some agents more weight in the preference ranking.

A **Social Welfare Function** is any form of aggregation that ensures that the societal "utility"function is monotonically increasing in the individuals utility function. As long as the welfare function is monotonically increasing in everyone's utility function, preferences will be consistently represented, and we can write it in a general form $W(u_1, ..., u_n)$ in an economy with n agents. In that case, the equal weighted sum above is one possible representation, and it is typically referred to as **a Classical Utilitarian or Benthamite** Welfare Function, and a generalized form of that idea is with unequal weights,

$$W(u_1, ..., u_n) = \sum_{i=1}^{n} \omega_i u_i$$

where ω_i are weights assigned to each constituent in the decision making process.

Yet another interesting aggregation is the **minimax** or **Rawlsian** Welfare Function:

$$W(u_1, ..., u_n) = min\{u_1, u_2, ..., u_n\}$$

which says that the welfare of a society is dependent soley on the worst off agent.

Essentially there is quite a myriad of ways to aggregate, and each represents a differing social ethical view. What would you do?

3 Welfare Maximization

Given a welfare function, the idea of welfare maximization is merely the a constrained maximization problem, where the objective is to maximize everyone's welfare using the Social Welfare Function as the objective, and the choices must be made under the constraint that any allocation must be feasible. That is

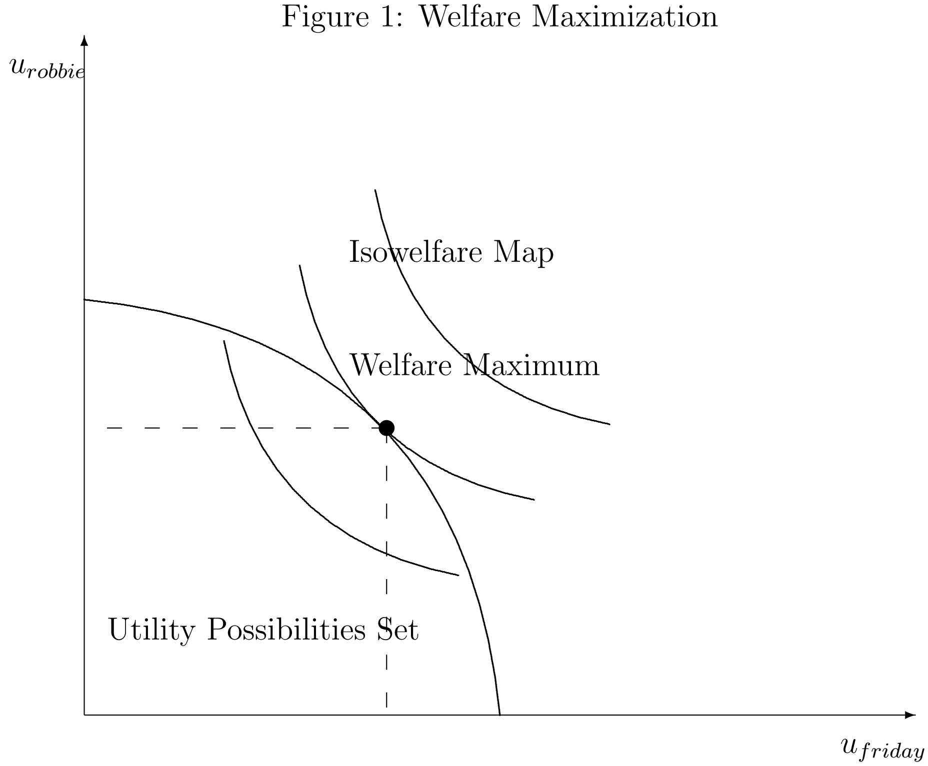
$$\max W(u_1, ..., u_n)$$

such that

$$\sum_{i=1}^{n} x_i^1 = X^1, ..., \sum_{i=1}^{n} x_i^k = X^k$$

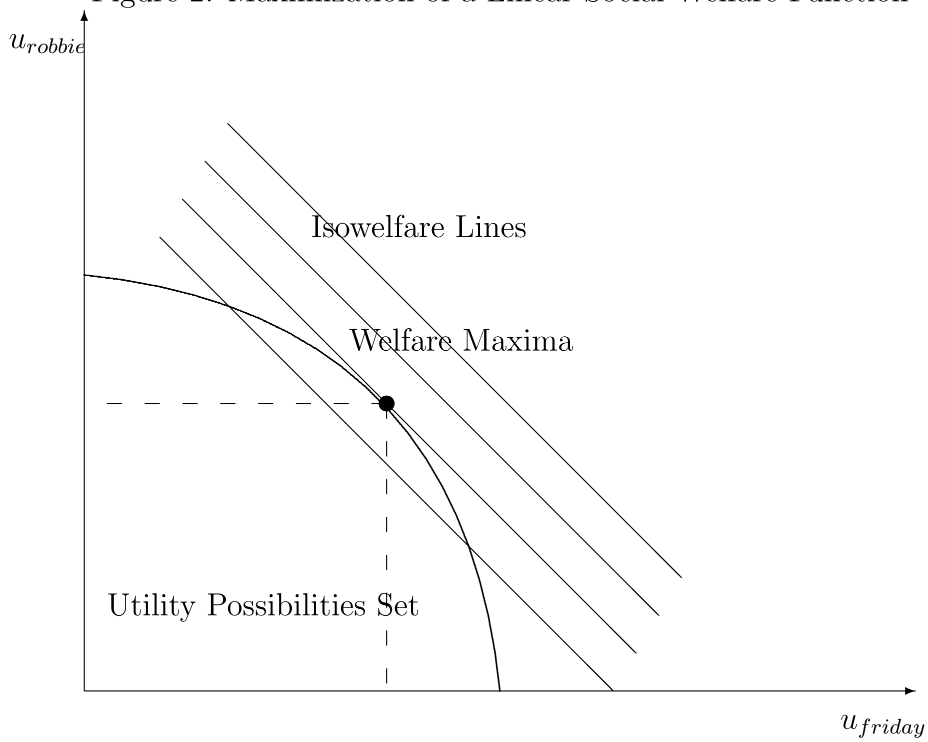
where x_i^j denotes the amounts of good j, allocated to individual i, in an economy with k goods to allocate, and where the society is endowed with X^j quantity of goods in total. A diagrammatic representation of the above problem is depicted below in a simple two agent society, and guess who's holding the party.

Figure 1: Welfare Maximization



The above diagram depicts the idea of welfare maximization. The utility possibilities set depicts the set of utility realizations that are derived from feasible allocations. While the isowelfare map is the set of aggregated utility curves. Note that since each individual's utility function is convex, then so to is their sum. The diagram below more accurately depicts a isowelfare map where the Social Welfare Function is a weighted sum of utility realizations.

Figure 2: Maximization of a Linear Social Welfare Function



It should be obvious that the optimal solution is a tangency between the utility possibilities frontier and the isowelfare line since it is only at tangency is the social welfare function the highest.

Further, this maximal welfare allocation is Pareto Efficient. Suppose it is not Pareto Efficient, so that we can find another feasible allocation that can make at least some parties better off without making any other constituent worse off. However, by definition a social welfare function is increasing in the utility measure of its constituents. This would imply that the welfare function must be above and beyond the utility possibility frontier, which would make that infeasible, contradicting the idea that the solution is yields the highest welfare measure.

4 Individualistic Social Welfare Functions

You might have been wondering whereas in consumer theory, we think of individuals having utility over their own consumption of goods, principally based on the implicit assumption that economic agents are selfish, here we have somehow veered away by supposing that agent care about entire allocations of goods to everyone (it has been found that an increase in your neighbours' wealth while holding yours constant, reduces your welfare, though the empirical exercise is totally flawed).

Well, we can similarly account for that fact and structure the Social Welfare Function as follows,

$$W = W(u_1(x_1), u_2(x_2), ..., u_n(x_n))$$

Then the welfare function is directly a function of the individuals' utility levels, but indirectly a function of individuals' consumption bundles, whereas before the relationship of the latter is direct. This form of utility function is known as a **Individualistic Welfare Function** or a **Bergson-Samuelson Welfare Function**. Further note that if instead, we think of the constituents as being households, then we can this of the above utility as that of a household, which in itself, is an aggregation within the household of the constituents individual utility functions.

Under this scenario, there are no consumption externalities, consequently the results from our previous discussion on general equilibrium in a pure exchange economy applies. That is a Pareto Efficient Equilibrium is a Competitive Equilibrium and vice versa (Recall the latter holds except when production, or the competitive market has increasing returns to scale). Consequently, we can say that a Welfare Maximum is a competitive equilibrium, and vice versa for some welfare function.

5 Fair Allocations

The advantage of using the social welfare function is its generality, however, the sacrifice in specificity means that it is devoid of any ethical judgment (though it must also be conceded that not all ethical judgments are ethical from varying ethical stand points). We can of course start from some specific moral judgment and examine their consequent implications.

The approach via **Fair Allocations** is to start from a "Fair "way of allocating a set of goods, and examine the implications of this allocation using what we have studied about general equilibrium. The simplest starting point would be one where you are tasked to divide a bundle of goods among individuals who are equally deserving under any standard. Suppose we are thinking of human ability, so that everyone is capable of enhancing the value of everyones' life, and has done so. The output has now been produced, and you the philosopher king who's wondering how you could perform the division. What would you do? Like most, given such an enviably simple task, the answer is equal division. Yes? That's the principal ideal behind the notion of fairness, perfect symmetry. No one could want what the other has since they're all exactly the same. However, if you really think about it, this allocation need not be Pareto Efficient given that individuals being equally deserving does not imply they have the same preferences, providing an impetus to trade with one another to achieve a higher level of utility. However, does this trade towards a Pareto Efficient allocation imply that the final allocation in society is (still) fair? In truth, if we are to think purely of trade as an exchange, then it is not necessarily true that the final allocation is fair. Consider a simple situation (noted in Hal Varian's book) that you have three citizens under your charge. A, B and C, where both A and B has the same preference system, and C is the weirdo. Suppose A meets with C accidentally as a nice little Bistro, strikes up a friendly conversation that lead to a trade agreement on account of their affection for each other. The trade leads them to the situation where they are both better off. Now B finds out from the grape vine, well it must be A bragging about his trade. Now B searches out C, but although C likes B just as much, she is unable to trade with B without making herself worse off since the last trade was Pareto Efficient with a type A individual which is exactly the same as B. In that case, the allocation that had evolved from an initially fair allocation does not end up being "fair"at the end of the day. This implies that arbitrary trading will not necessarily leads to a fair allocation. If you are reading between the lines, you would have taken the hint that I am alluding to the Competitive mechanism where we use prices. If you haven't think harder next time.

6 Envy and Equity

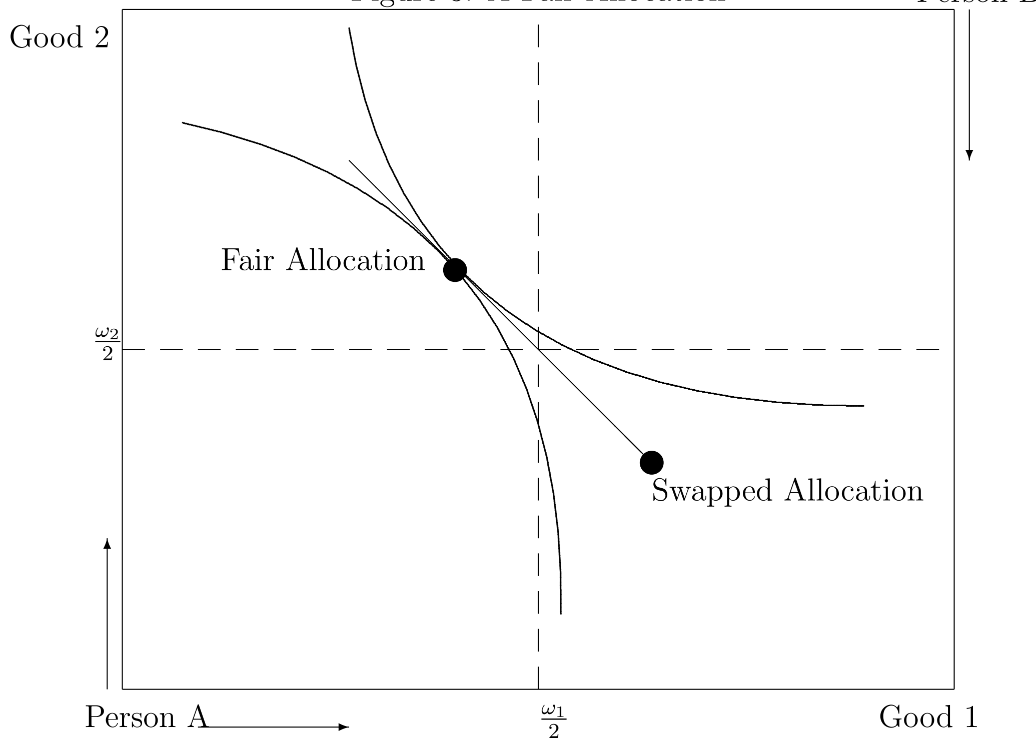
You might be thinking that I pulled a fast one on you since I hadn't defined for you what is a symmetric and an equitable allocation. Well that what we will do now.

1. An allocation is **Equitable** if no agent prefers any other agent's bundle of goods to his or her own. If agent i prefers the allocation of agent j, that we say that i envies j. Note we are not measuring envy based on the agent's level of utility, but purely on the allocation. So that it is possible to have an agent not feel envy in the above terms, but if she knew how much more happy the other party was, she'd feel a sense of envy in the most conventional human sense. The latter is not what we are talking about.

2. If an allocation is both **Equitable** and **Pareto Efficient**, we say that the allocation is **Fair**.

Let us begin where we left off, where the the magnanimous philosopher king or queen decides to start everyone on an equal footing in his kingdom of equals with his two loyal subjects sharing the endowment of two goods, ω_1 and ω_2 equally, bringing the start of this story to smack right in the middle of our Edgeworth Box. He allows them to set their desired prices to reflect the true value of the riches of his or her kingdom, so that they trade eventually to the Pareto Efficient Equilibrium depicted in figure 3, and we know that that is a competitive equilibrium as well. We can check whether this final allocation is indeed equitable by performing a counter factual by allocating one of their allocations to the other and vice versa to see if the new allocation would give rise to both of them being on a higher indifference curve. Obviously, from figure 3, we see that if they had swapped their allocation, they would both have been worst off, which means the initial Pareto Efficient Allocation is a equitable allocation since there is no element of envy involved. Consequently, by our above definition, we have shown that this Pareto Efficient and Equitable Allocation started from a equal division of societal endowment is indeed Fair, and the Philosopher king or queens is happy. Note that this equilibrium could just as well have been achieved through trading without a competitive market mechanism involved.

Figure 3: A Fair Allocation



A Competitive Equilibrium from equal division is a Fair allocation.

Can we show more formally that the competitive equilibrium derived from an equal division of endowment is equitable?

Suppose not, so that A envies B. Then the following must be true,

$$\left(x_A^1, x_A^2\right) \prec_A \left(x_B^1, x_B^2\right)$$

But by definition that the original bundle was the competitive equilibrium, i.e. the best that A can do given his income, it would mean that B's allocation is not affordable to him,

$$p_1 x_B^1 + p_2 x_B^2 > p_1 \omega_A^1 + p_2 \omega_A^2 = p_1 \omega_B^1 + p_2 \omega_B^2$$

However, since they had begun from an equal division allocation, their incomes would have been the same, and the above inequality means that neither could B have afforded her bundle, which contradicts the competitive allocation was achieved! Consequently, beginning from an equal division, a Competitive Equilibrium (and is Pareto Efficient) is a Fair Allocation.

It should nonetheless be noted that the equal division is dependent on all being deserving, the determination of which may be difficult if not potentially controversial.