Mathematical Economics, ECON 471, Lecture 7
Solving Simple Ordinary Differential Equations

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In general, most of the dynamic models developed in economics do not have close form solution. Nonetheless, should our solutions generate dynamic equations that can be solved with some ease, we should be able to recognize and solve them. To this extent, we will examine linear differential equations.

1 First Order Differential Equations

1.1 First Order Homogeneous Differential Equation

Structure of such a differential equation is,

$$y' + ay = 0 \tag{1}$$

The solution to such an equation is,

$$y(t) = Ae^{-at} \tag{2}$$

The reason is because of the following,

$$\begin{aligned} y' + ay &= 0 \\ \Rightarrow \frac{y'}{y} + a &= 0 \\ \therefore \ln y &= -at + \alpha \\ \Rightarrow y &= e^{-at}e^{\alpha} \\ &= Ae^{-at} \end{aligned}$$

We refer to this as the Complementary Function. Note the A is solved with the use of the initial state.

1.2 First Order Non-Homogeneous Differential Equation

The structure of such an equation is,

$$y' + ay = b \tag{3}$$

Here the Complementary Function remains the same.

$$y(t) = Ae^{-at} \tag{4}$$

However, now we have another part of the solution to contend with, the right hand side of the equation. The solution to that is known as the Particular Integral, and it is just,

$$y_p = \frac{b}{a} \tag{5}$$

Finally, the complete solution is,

$$\begin{aligned} y(t) &= (A - y_p)\, e^{-at} + y_p \\ &= \left(A - \frac{b}{a}\right) e^{-at} + \frac{b}{a} \end{aligned}$$

When $a = 0$, the differential equation is,

$$y' = b \tag{6}$$

so that the solution is simply the integral of the equation with respect to t

$$y(t) = bt + \alpha \tag{7}$$

1.3 First Order Non-Homogeneous Differential Equation with Variable Coefficient & Variable Term

The general form of a first order non-homogeneous linear differential equation is of the form,

$$y' + u(t)y = w(t) \tag{8}$$

where $u(t)$ and $w(t)$ are the variable coefficient and variable term respectively.

1.3.1 Homogeneous Case

In the homogeneous case, $w(t) = 0$, then equation (8) becomes,

$$\begin{aligned}
y' + u(t)y &= 0 \\
\Rightarrow \frac{y'}{y} &= -u(t) \\
\Rightarrow \int \frac{y'}{y} &= \int -u(t)dt \\
\Rightarrow \ln y + c &= \int -u(t)dt \\
\Rightarrow \ln y &= -c - \int u(t)dt \\
\Rightarrow y &= e^{-c-\int u(t)dt} \\
\Rightarrow y &= Ae^{-\int u(t)dt}
\end{aligned}$$

Note that this is a general solution, and subsumes our solution to the homogeneous case in the previous section since that solution can be written as,

$$\begin{aligned}
y &= e^{\alpha - at} \\
&= e^{\alpha}e^{-at} \\
&= Ae^{-\int (a)dt}
\end{aligned}$$

To get an idea of how this works, consider the following example:

Example 1 *Given,*

$$y' + 3t^2 y = 0$$

Then $u(t) = 3t^2$, *so that* $\int 3t^2 dt = t^3 + \alpha$, *so that the solution is,*

$$y = Ae^{-t^3}e^{-\alpha} = Be^{-t^3}$$

1.3.2 Non-Homogeneous Case

In the non-homogeneous case, the differential equation will be of the form,

$$y' + u(t)y = w(t)$$

The solution to this case is,

$$y(t) = e^{-\int (u)dt}\left(A + \int we^{\int (u)dt}dt\right)$$

The rationale for this solution will be discussed below for all the prior cases. To understand their use, we will go through some examples.

Example 2 *Consider the following differential equation,*

$$y' + 2ty = t$$

Comparing to the formula, we have $u(t) = 2t$*, which then implies that* $\int u(t)dt = t^2 + \alpha$*. Next, note that* $w(t) = t$*. Therefore, applying the formula,*

$$\begin{aligned} y(t) &= e^{-t^2-\alpha}\left(A + \int te^{t^2+\alpha}dt\right) \\ &= e^{-t^2-\alpha}A + e^{-t^2-\alpha}\left(\frac{e^{t^2+\alpha}}{2} + \beta\right) \\ &= e^{-t^2}\left(e^{-\alpha}A + e^{-\alpha\beta}\right) + \frac{1}{2} \\ &= Be^{-t^2} + \frac{1}{2} \end{aligned}$$

You may check the solution by finding y' *and substituting the derivative back into the differential equation.*

Indeed, if you had ignored all the constants in the integration process, and directly applied the formula, you would still obtain the same solution. To see that, consider the next example.

Example 3 *Consider now,*

$$y' + 4ty = 4t$$

so that $u(t) = 4t$ *which implies* $\int u(t)dt = 2t^2$*. As well,* $w(t) = 4t$*, which in turn implies that* $\int 4te^{2t^2}dt = e^{2t^2}$*. So that applying the formula, we have the following solution,*

$$y(t) = e^{-2t^2}\left(A + e^{2t^2}\right)$$

From this discussion, it should also become clear how we obtain our previous formulas when $u(t)$ and $w(t)$ are constants, instead of being dependent on t. In case it is unclear, let $u(t) = a$ and $w(t) = b$, so that $\int adt = at$, and $\int be^{at}dt = \frac{b}{a}e^{at}$. Now applying the formula, we have

$$\begin{aligned} y(t) &= e^{-at}\left(A + \frac{b}{a}e^{at}\right) \\ &= Ae^{-at} + \frac{b}{a} \end{aligned}$$

2 Exact First Order Differential Equations

To understand how and when a first order differential equation can be solved, first note that we can think of the entire equation as a function of y and t. Secondly, although the problems we work with are linear in nature, the same ideas work for non-linear differential equations.

Next note that in general, a function of the two variables y and t can be written as $F(y,t)$, so that its total derivative is,

$$dF(y,t) = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt$$

and when the above equation is equal to zero, it is what is known as an *exact differential equation*. In general, a differential equation can be generalized to be of the form,

$$\Theta dy + \Gamma dt = 0$$

and it is considered *exact* if and only if there exists a function $F(y,t)$ such that $\Theta = \frac{\partial F}{\partial y}$ and $\Gamma = \frac{\partial F}{\partial t}$. You will recall that based on Young's Theorem,

$$\begin{aligned} \frac{\partial^2 F}{\partial y \partial t} &= \frac{\partial^2 F}{\partial t \partial y} \\ \Rightarrow \frac{\partial \Theta}{\partial t} &= \frac{\partial \Gamma}{\partial y} \end{aligned}$$

This last equation then allows us to test whether the solution to our differential equation is correct or otherwise, or a test of the exactness of our differential equation. Put another way, if the left and right hand side of the last equality are the same, we have an exact differential equation, and the method of solution to be described below follows. Further since there are no restrictions placed on y, it is possible to have nonlinear (in y) differential equations offering exact solutions.

The import of these findings above, is that since exactness implies

$$\begin{aligned} dF(y,t) &= 0 \\ \Rightarrow F(y,t) &= \beta \end{aligned}$$

Next, observe that since $\Theta = \frac{\partial F}{\partial y}$, this then implies that the function $F(y,t)$ must have the integral of Θ with respect to y. In other words,

$$F(y,t) = \int \Theta dy + \xi(t)$$

where $\xi(t)$ is the equivalent of a constant. The reason the second term on the right hand side is added is because in partially differentiating $F(y,t)$, t or any function of t would have been treated like a constant, so that in "reverse engineering" the function $F(y,t)$, we would have to reinstate it back. The next question is how we can backout the value of functional form of $\xi(t)$? Observe that,

$$\begin{aligned}\frac{\partial F(y,t)}{\partial t} &= \frac{\partial\left(\int \Theta dy\right)}{\partial t} + \xi'(t)\\ &= \Gamma\end{aligned}$$

This then suggest that to solve the exact differential equation, we would need to use the idea behind $\Gamma = \frac{\partial F}{\partial t}$.

The precise steps are as follows:

1. Reduce the differential equation to the form,

$$F(y,t) = \int \Theta dy + \xi(t)$$

and relate to

$$\frac{\partial F(y,t)}{\partial y}dy + \frac{\partial F(y,t)}{\partial t}dt = 0$$
$$\Rightarrow \Theta(y,t)dy + \Gamma(y,t)dt = 0$$

Next realize that $\Theta \equiv \Theta(y,t)$.

2. Denote $\int \Theta dy = \theta(y,t)$, so that

$$\begin{aligned}\frac{\partial F(y,t)}{\partial t} &= \frac{\partial \theta(y,t)}{\partial t} + \xi'(t)\\ &= \Gamma(y,t)\end{aligned}$$

where $\Gamma \equiv \Gamma(y,t)$. We can now compare $\Gamma(y,t)$ to $\frac{\partial\theta(y,t)}{\partial t} + \xi'(t)$. This will give us,

$$\xi'(t) = \gamma(t)$$

Notice that the right have side, $\gamma \equiv \gamma(t)$ since otherwise, a derivative with respect to y would yield an output.

3. Thus,

$$\xi(t) = \int \gamma(t)dt = \zeta(t)$$

noting that you can ignore the constant after integration.

4. Then combining all the results, we have

$$F(y,t) = \theta(y,t) + \zeta(t) = \nu$$

It should be noted that just because we do not have an exact differential equation does not imply we cannot solve the differential equation. If we can find a common factor such that by multiplying it through the entire equation, we create a exact differential equation, we would still be able to solve it. This common factor is known as an *integrating factor*.

2.1 Solution of First Order Linear Differential Equation

The general form of a first order linear differential equation is

$$\frac{dy}{dt} + u(t)y = w(t) \tag{9}$$

$$\Rightarrow dy + (u(t)y - w(t))dt = 0 \tag{10}$$

It is easy to see that in the current form, the differential equation is inexact. However, there is an integrating factor. Let ϕ be the integrating factor so that,

$$\phi dy + \phi(u(t)y - w(t))dt = 0$$

We know that for this to work, we need

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi(u(t)y - w(t))}{\partial y}$$

Since both $u(t)$ and $w(t)$ are functions of t only, it would be prudent and simple to look for a $\phi \equiv \phi(t)$, in other words, to look for a function ϕ that is likewise a function of t alone as well. In that situation, the test of exactness reduces to,

$$\begin{aligned}
\frac{\partial \phi(t)}{\partial t} &= \phi(t)u(t) \\
\Rightarrow \frac{\frac{\partial \phi(t)}{\partial t}}{\phi(t)} &= u(t) \\
\Rightarrow \ln \phi(t) &= \int u(t)dt + \alpha \\
\Rightarrow \phi(t) &= e^{\alpha} e^{\int u(t)dt}
\end{aligned}$$

However, the constant is inconsequential, and we can set $\phi(t) = e^{\int u(t)dt}$. This then means that,

$$e^{\int u(t)dt} dy + e^{\int u(t)dt}(u(t)y - w(t))dt = 0$$

Adopting the four step procedure from the previous section,

1. First integrating $\Theta(y,t)$ with respect to y, we get

$$F(y,t) = \int e^{\int u(t)dt} dy + \xi(t) \quad = \quad ye^{\int u(t)dt} + \xi(t)$$

2. This is followed by the differentiation of $F(y,t)$ with respect to t

$$\frac{\partial F}{\partial t} = yu(t)e^{\int u(t)dt} + \xi'(t)$$

Next notice that,

$$\Gamma(y,t) = e^{\int u(t)dt}(u(t)y - w(t))$$

Therefore,

$$\xi'(t) = -w(t)e^{\int u(t)dt}$$

3. Next, we find $\xi(t)$,

$$\xi(t) = -\int w(t)e^{\int u(t)dt} dt$$

4. Combining all the parts, we have

$$\begin{aligned} F(y,t) &= ye^{\int u(t)dt} - \int w(t)e^{\int u(t)dt} dt \\ \Rightarrow \quad & ye^{\int u(t)dt} - \int w(t)e^{\int u(t)dt} dt = \nu \end{aligned}$$

Therefore,

$$\begin{aligned} ye^{\int u(t)dt} &= \nu + \int w(t)e^{\int u(t)dt} dt \\ \Rightarrow y(t) &= e^{-\int u(t)dt}\left(\nu + \int w(t)e^{\int u(t)dt} dt\right) \end{aligned}$$

2.2 First Order Nonlinear Differential Equations

As you may have wondered, what if the differential equation were nonlinear? Is there anyway of finding a solution, and under what circumstances is this possible. A first order nonlinear differential equation is of the form,

$$f(y,t)dy + g(y,t)dt = 0 \tag{11}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{g(y,t)}{f(y,t)} = h(y,t) \tag{12}$$

There are three types of cases we can consider:

1. When the *differential equation is exact*, we can again use the four step method we had discussed previously.

2. When the differential equation is additively *separable* in the following sense,

$$f(y)dy + g(t)dt = 0 \tag{13}$$

When this is the case, all we need to do is to integrate both sides independently. Of course, the usual caveat is that for this to work, both sides should yield close form solutions.

3. The final case when we can solve the nonlinear system is when the differential equation is reducible into a linear form. Consider the general nonlinear differential equation of $\frac{dy}{dt} = h(y,t)$ taking on the following form,

$$\frac{dy}{dt} + \rho y = \tau y^{\nu} \tag{14}$$

where $\rho \equiv \rho(t)$, $\tau \equiv \tau(t)$, and ν is any number not equal to 0, or 1. In this case, the equation is known as a *Bernoulli Equation*. When the nonlinear differential equation is a Bernoulli Equation, then it can always be reduced to a linear differential equation, so that the previous techniques would apply.

The reduction process is as follows:

   (a) Divide the differential equation through out by y^{ν} so that we get,

$$y^{-\nu}\frac{dy}{dt} + \rho y^{1-\nu} = \tau$$

   (b) Let $z = y^{1-\nu}$, so that by the chain rule $\frac{dz}{dt} = \frac{dz}{dy}\frac{dy}{dt} = (1-\nu)y^{-\nu}\frac{dy}{dt}$. This then imply that,

$$\begin{aligned} \frac{1}{1-\nu}\frac{dz}{dt} + \rho z &= \tau \\ \Rightarrow dz + \left[(1-\nu)\rho z - \tau(1-\nu)\right] dt &= 0 \end{aligned}$$

   Therefore, now we can apply the techniques we have discussed thus far.

   (c) Find $z(t)$, and transform the solution back to obtain $y(t)$.

3 Second Order Differential Equation

3.1 Second Order Homogeneous Differential Equation

The structure of such an equation is,

$$y'' + a_1 y' + a_2 y = 0 \tag{15}$$

Here the Complementary Function is usually obtained by adopting a trial solution of $y = e^{rt}$, which implies that

$$\begin{aligned} y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

So that the differential equation looks like,

$$\begin{aligned} r^2 e^{rt} + a_1 r e^{rt} + a_2 e^{rt} &= 0 \\ r^2 + a_1 r + a_2 &= 0 \end{aligned}$$

We can then use the coefficients from the differential equations to solve for the characteristic roots.

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

The allocation of the roots is not of importance. Therefore the Complementary Function is (**Note that the exponent does not have a negative sign**),

$$A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{16}$$

The idea behind this is that we are in fact guessing that the solution is of the form αe^{rt}. In doing so, we have implicitly accepted that $y' = \alpha r e^{rt}$, and $y' = \alpha r^2 e^{rt}$, which would give,

$$\begin{aligned} y'' + a_1 y' + a_2 y'' &= 0 \qquad (17) \\ \alpha r^2 e^{rt} + \alpha a_1 r e^{rt} + \alpha a_2 e^{rt} &= 0 \\ \left(r^2 + a_1 r + a_2\right) \alpha e^{rt} &= 0 \end{aligned}$$

It must be kept in mind that any complementary function solution has to satisfy 17. The next question is when wouldn't our guess work?

The above solution works if the roots are distinct. In the case when we have repeated roots, that is $r_1 = r_2 = r$. In this case, the suggested complimentary function become,

$$A_1 e^{r_1 t} + A_2 e^{r_2 t} = (A_1 + A_2)e^{rt} = A_3 e^{rt}$$

This then implies that we need to find another portion to the complementary function. This is because in differentiating the solution to obtain the differential equation, we would have lost two constants, in consequence, when we reverse the process to find the solution, we need to recoup the two constants for a second order differential equation.

The additional guess so as to obtain a second constant for the complementary function needs to be linearly independent from the original guess, failing which, we would arrive at the original problem again, since there is only one root. A natural guess would be $\beta t e^{rt}$. Then this would in turn imply that,

$$\begin{aligned} y' &= \beta e^{rt}(1 + rt) \\ y'' &= \beta e^{rt}(2r + r^2 t) \end{aligned}$$

so that when we substitute this back into the original differential equation, we have,

$$\begin{aligned} \beta e^{rt}(2r + r^2 t) + \beta a_1 e^{rt}(1 + rt) + \beta a_2 t e^{rt} &= 0 \\ \beta e^{rt}\left((2r + r^2 t) + a_1(1 + rt) + a_2 t\right) &= 0 \end{aligned}$$

This is a valid solution since both the original guess, and the new guess has the root r, they will both be zero, thus not impinging on the differential equation. Therefore, in the case when we have a distinct root, the complementary function is of the form,

$$A_1 e^{rt} + A_2 t e^{rt}$$

At this juncture, it is an opportune moment to discuss dynamic stability. Dynamic stability occurs only when the complementary function tends towards zero, as time t tends towards ∞, or in short, $y_c \to 0$, as $t \to \infty$. When this occurs, then as time proceeds, the path would eventually reach the particular integral, y_p! The only way this can happen is the case when we have two roots is if and only if both the roots are negative, since as $t \to \infty$ $e^{rt} \to 0$. However, should any one of the roots be positive, the positive portion of the complementary function, y_c, would dominate as $t \to \infty$ (It should also be clear now why we have a negative sign in our complementary function for the first order differential equation.) This condition likewise holds for the distinct root case when $r < 0$, since for $r < 0$

$$\begin{aligned} \lim_{t\to\infty} te^{rt} &= \lim_{t\to\infty} \frac{t}{e^{-rt}} \\ &= \lim_{t\to\infty} \frac{1}{-re^{-rt}} = 0 \end{aligned}$$

Therefore, as long as $r < 0$, the solution is dynamically stable.

3.2 Second Order Non-Homogeneous Differential Equation

As in the first order case, we have to contend with the Particular Integral when the differential equation is non-homogeneous.

$$y'' + a_1 y' + a_2 y = b \tag{18}$$

There are several cases we need to contend with. When a_1 and a_2 are non-zero (**Note that there is no t for the particular integral**),

$$y_p = \frac{b}{a_2} \tag{19}$$

When $a_2 = 0$,

$$y_p = \frac{b}{a_1} t \tag{20}$$

The solution is just the sum of the Complementary Function and the Particular Integral. Note that it is unlike the first order case.

$$y(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + y_p \tag{21}$$

When both a_1 and a_2 are zero,

$$\begin{aligned} y'' &= b \\ \Rightarrow y' &= bt + \alpha \\ \Rightarrow y(t) &= \frac{b}{2} t^2 + \alpha t + \beta \end{aligned}$$

The particular integral here is $\frac{b}{2} t^2$.