Economists typically assume that firms or a firm's owners try to maximize their profit. Let R be revenues of the firm, and C be the cost of production, then a firm's profit can be represented as follows,

$$\Pi = R - C$$

To be precise, since revenues, and costs are dependent on the level of production, or the amount of goods a firm produces, and the revenues are dependent on how much a firm sells its output, we can rewrite the relationship as follows,

$$\Pi = PQ - C(Q)$$

Where $C(Q)$ reads as "cost is a function of level of output".

Production

A firm uses technology or production process in concert with inputs or factors of production to produce output which is sold. These inputs can be broadly categorized as

1. Capital: Long lived inputs such as land, buildings, and equipment
2. Labor: Human services
3. Materials: Raw Goods

The output produced can be either services rendered or goods for consumed, or goods used in other production processes to produce other goods, or services.

Using this information, we then write the output of a firm as

$$Q = F\left(L, \overline{K}\right)$$

which reads as "output is a function of Labor and Capital", and implies that output is dependent on the production process which uses Labor (L) and Capital (K) to produce its output, which may or may not be a final good or service. We write K with a bar on top, $\overline{K}$, to denote the fact that typically in a production process, the level of capital is fixed. $F(.)$ is then called the production function, and describes how a firm can combine the inputs, here just L, and K, to produce their goods, or services.

Subsequently, we will be examining the degree of productivity of factors of production. We will now highlight some of the common concepts,

1. ***Marginal Product of Labor*** measures the change in total output as a result of an additional unit of labor. The parallel concept for capital is just the ***Marginal Product of Capital***, and measures the change in total output as a result of an additional unit of capital.

$$MP_L = \frac{\Delta Q}{\Delta L}$$

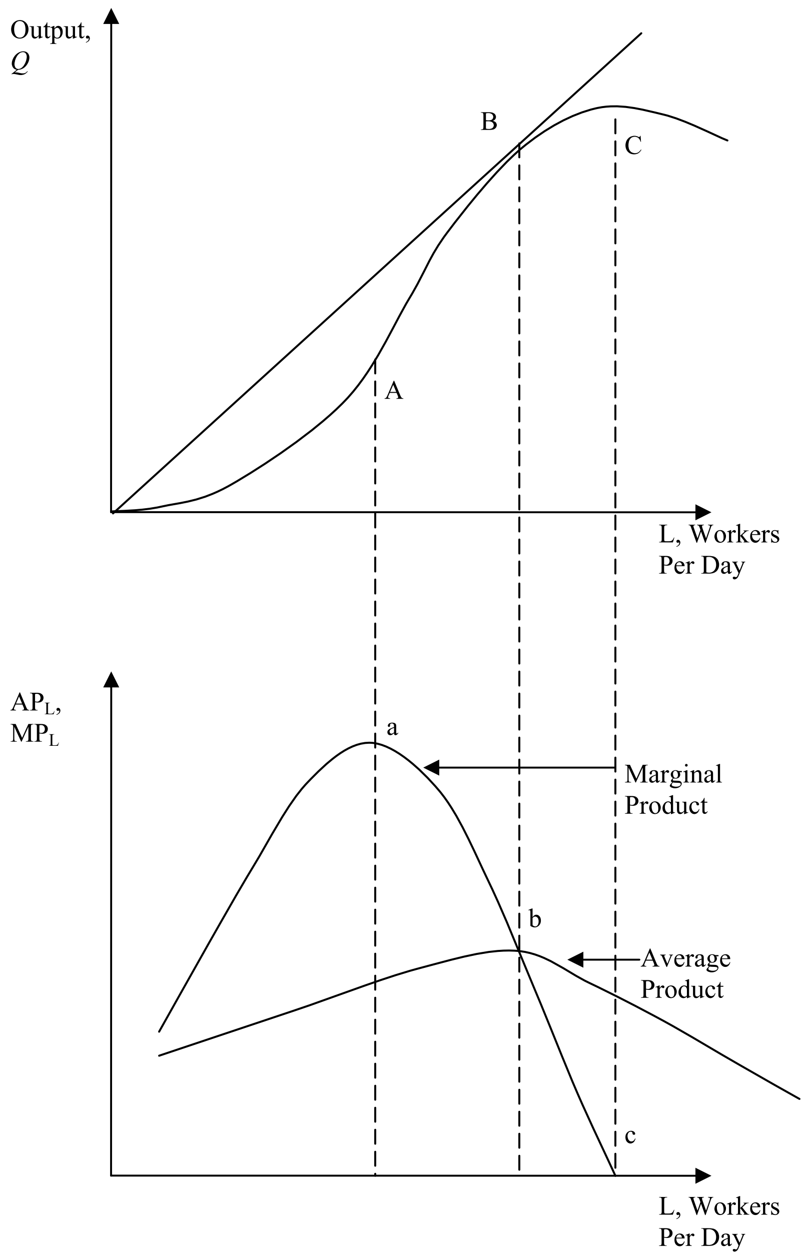
$$MP_K = \frac{\Delta Q}{\Delta K}$$

2. ***Average Product of Labor/Capital*** is just the ratio of output to the number of workers/units of capital used to produce the output.

$$AP_L = \frac{Q}{L}$$

$$AP_K = \frac{Q}{K}$$

These are represented diagrammatically below,



The first diagram depicts the how total output who change as the number of the variable input, labor rises. The second diagram relates how different segments of the total product line relate to the different measures of average and marginal product of labor. Similar relationships can be drawn in terms of capital or the fixed input of production.

First note that the total product curve is first convex, and then concave. That is the slope of the total product curve first increases as more labor is used, and then begins to fall. In fact at point C, the slope is zero. Beyond point C, the additional labor force does not yield any increase in output, but creates a fall in output. The reason is due to the fact that as output increases as a result of an increase in the factor input, labor, the additional output of the additional labor yields less output produced, that is the marginal product of labor falls with the additional labor. This is known as the ***Law of Diminishing Marginal Returns***. Note that the diagram is drawn given a particular level of capital or any other inputs that may be used in the production process. If any of those variables changes, the entire Total product curve would change.

If we draw a curve for the relationship between the marginal product of labor, MP_L, and the quantity of labor, the point where the total product curve changes from a convex curve to a concave on corresponds with the peak, since all point beyond A corresponds with a lower level of marginal product of labor. It should be noted that the marginal product of a factor input is the slope of the total product curve. The average product of labor, or for any factor input, is the line that extends from the origin to that point on the total product curve. We can similar plot this measure on the same graph as $MP_{L,}$ which have drawn above for labor, and is labeled AP_L. When AP_L corresponds to the slope of the total product curve at that point, it is the point where the AP_L and MP_L intersect, point b on the graph above. The point on the total product curve starts falling is point where the MP_L becomes negative, point c above. It must be noted that the idea of diminishing marginal returns occurs when we are holding the other variable, such as other factor inputs, and technology constant. It is thus possible that as we increase all factor inputs, that we may never see the incidence of diminishing marginal product of the factor of input.

Isoquants

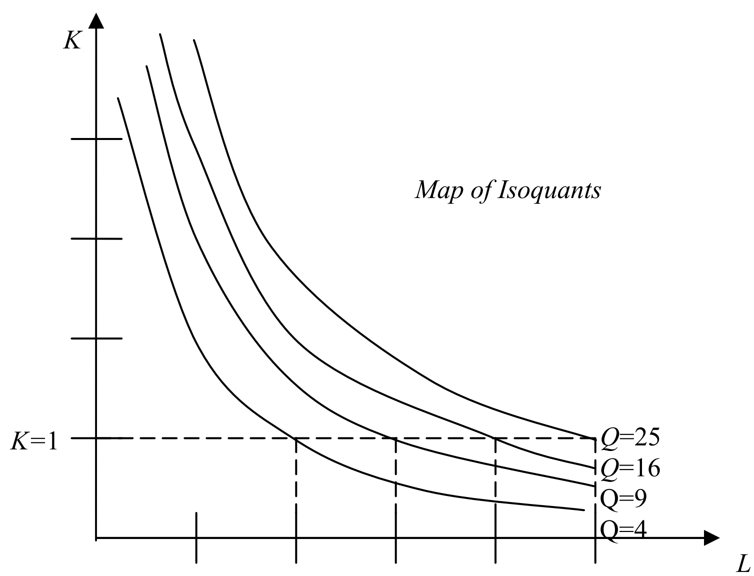
We have examined the manner in which output is raised as we increase factor input holding some other factor inputs constant. This is principally focusing on the impediment in the short run where we typically think of some types of factor inputs being unalterable, such as capital like machines, or factory building, or land. In the long run, the firms can change all factor inputs and technologies used. How do we then depict this idea? We can take a hint from the manner in which we have examined the individuals' choice. Consider the technology that goes toward the production of a good as a Cobb Douglas function such as the following, so that the quantity of output produced in terms of two inputs, say labor (L) and capital (K) can written as follows,

$$Q = K^2 L^2$$

The table below gives the output that could be produced of this product with such a technology and various combinations of inputs.

		Quantity of Labor					
		1	**2**	**3**	**4**	**5**	**6**
Quantity Capital	**1**	Q=1	4	9	16	25	36
	2	4	16	36	64	100	144
	3	9	36	81	144	225	324
	4	16	64	144	256	400	576
	5	25	100	225	400	625	900
	6	36	144	324	576	900	1296

Based this technology, you can see that for various combinations of inputs, we could in fact produce the same level of output, just as we depicted in our discussion of individual choices. Diagrammatically, we can note some of these combinations. This would then produce a map not unlike the indifference map, with the exception that instead of each curve describing a level of utility, each curve here depicts the value of output that could be produced with those combinations of inputs.



From the same diagram above of the isoquant map, we can see the marginal output of labor by holding the level of capital fixed at a particular level, above depicted as K=1. (You should note that this is why we draw Cobb-Douglas functions the way we do, as convex curves). In fact the marginal output of the additional labor is 5, 7, 9, moving from 2 workers, to 3, and from 3 to 4 and from 4 to 5 respectively. However, with such a Cobb Douglas function, with the above exponents, we will not see diminishing marginal returns since the function is strictly increasing. Can you show yourself? We will see decreasing marginal returns only is the exponents are below 1 individually. In fact if the sum of the two exponents sum to 1, with both being between 0, and 1 would guarantee we would have decreasing marginal returns of labor.

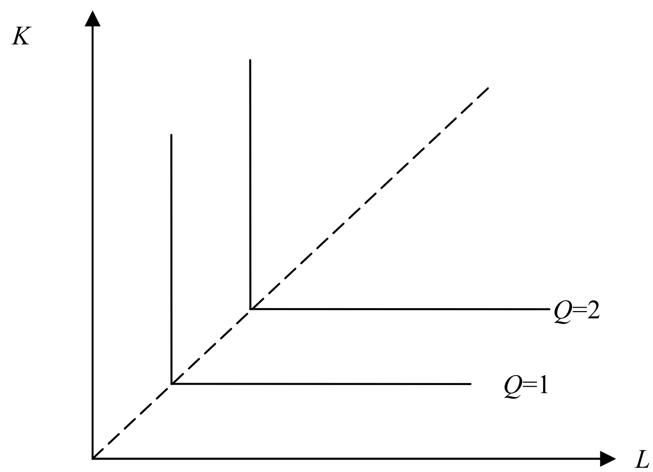
Properties of the Isoquant

1. The greater the Isoquant is from the origin, the greater is the level of output as you can see from the above diagram.
2. Just like indifference curve, isoquants do not cross. This is principally because firms are assumed to always be inefficient in its production, that is it will always choose to produce more output if it could, rather than producing less.
3. Isoquant slope downward.

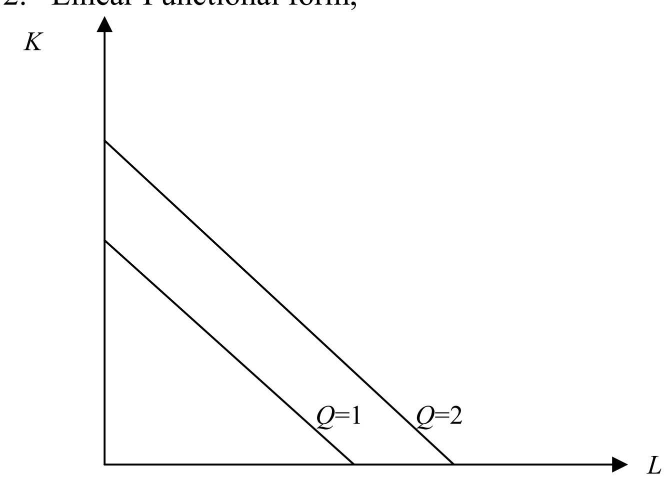
Shape of Isoquants

The shape we considered depending on the functional forms of indifference curves apply here again. Principally, the shape is dependent on our functional form assumption. So if we assume that the production technology follows

1. Leontieff Functional form,



2. Linear Functional form,



3. Cobb-Douglas form, is as first diagram.

This by no means is the above an exhaustive depiction, but most other functional forms would fall somewhere between the above.

Marginal Rate of Technical Substitution

Just as the slope of the indifference curve in our consumer theory analysis can be transformed into the ratio of the marginal utility of consumption of the respective goods, so to we can express the slope of the isoquant as the ratio of the marginal output of the factor of inputs. The parallel name for this to the consumer theory (marginal rate of substitution) is the marginal rate of technical substitution (*MTRS*). The *MRTS* tells us the number of units of capital the firm can replace with labor, while holding output constant. Further note that because the isoquant is downward sloping, the *MRTS* is negative. To be precise the proof of the relationship is as follows, and relies on the fact that as we move along the isoquant, there is in fact no change in the quantity of output. Suppose the production function is $Q = F(L, K)$. Then differentiating the equation totally, and noting the zero change in quantity, we get,

$$dQ = \frac{\partial F(L,K)}{\partial L} dL + \frac{\partial F(L,K)}{\partial K} dk = 0$$

$$\Rightarrow \frac{\partial F(L,K)}{\partial L} dL = -\frac{\partial F(L,K)}{\partial K} dK$$

$$\Rightarrow -\frac{\frac{\partial F(L,K)}{\partial L}}{\frac{\partial F(L,K)}{\partial K}} = \frac{dK}{dL}$$

$$\Rightarrow -\frac{MP_L}{MP_K} = \frac{dK}{dL} = \frac{\Delta K}{\Delta L} = MRTS$$

Just as in consumer theory, where as we move downward along an indifference curve, we experience diminishing marginal rate of substitution, so too as we move downward along a isoquant, we will see diminishing marginal rate of technical substitution. This means that as the firm reduces the use of one input, eventually, because both inputs are vital to production (which is dependent on the production technology as described by the production function), the less the firm uses of it, the more the firm needs to augment the forsaken input with the other input. Essentially, it is technology that will determine the *MRTS*.

Returns to Scale

The above discussion concerns the change in output with a change in input holding the others constant. We will now consider the change in output as a result of a proportion change in all inputs. This pertains to the idea of how effective, inputs are used on the whole as we increase all their proportions, and is largely dependent on technology as well. You may recall that we call these examinations ***economies of scale***. Consider a similar general production as before, $Q = F(L, K)$. When an increase in L and K by α,

1. Leads to an increase in Q by α as well, we say that the technology exhibits ***constant returns to scale***, i.e. $\alpha Q = F(\alpha L, \alpha K)$. Put another way, what this means is that a doubling of inputs, leads to a doubling of output.

2. Leads to an increase in Q by greater than α, we say that the technology exhibits ***increasing returns to scale***, i.e. $\alpha Q > F(\alpha L, \alpha K)$. Here, a doubling of inputs leads to a more than doubling of output.
3. Leads to an increase in Q by less than α, we say that the technology exhibits ***decreasing returns to scale***, i.e. $\alpha Q < F(\alpha L, \alpha K)$. A doubling of inputs leads to a less than doubling of outputs.

Costs

So far, we have discussed the production aspect of a firm. We have not considered the cost. The operation of a business involves costs, since no man is an island, and we will need machines, and other specialized skills to assist the operation. This then mean cost, since no one is going to offer their labor services or capital to run the business for free. Would you? Just as in consumer theory, the individuals maximize their welfare, or utility subject to their budget constraint as determined by their earned and unearned income, so too the firm's operation is not a constant stream of revenues only, and the firm maximizes their revenues subject to the cost of operation. Specifically, the firm maximizes their profit. Before we begin with that, we will briefly describe the cost function for the firm.

Just as in consumer theory where it is the sum of the value of goods individuals consume. The cost is simply just the sum of the value of inputs used in the production process. Let the inputs used in the production process be indexed by the subscript i, and let x_i be the quantity of inputs used. Further let the price of a unit of factor input be c_i. Therefore the cost of each set of factor input is $c_i x_i$. Let there be N inputs. Then the cost of a firm in its production process is,

$$c_1 x_1 + c_2 x_2 + \ldots = \sum_{i=1}^{N} c_i x_i$$

If we further subdivide within this set factor inputs into variable and fixed inputs, denoting the latter by a over-bar, letting the first n factor inputs be variable, then the above becomes,

$$\sum_{i=1}^{N} c_i x_i = \sum_{i=1}^{n} c_i x_i + \sum_{j=N-n+1}^{N} c_j \bar{x}_j$$

In a two factor world such as we have been talking about, where K is fixed in the short run, and L is variable, letting w denote wages of labor, and r as the rental paid to capital, a simple cost equation is,

$$Cost = wL + rK$$

Therefore, for a two factor firm we can rewrite a firm's short run profit as

$$\Pi = PQ - \left(wL + r\overline{K}\right) = \left(P \times F\left(\overline{K}, L\right)\right) - \left(wL + r\overline{K}\right)$$

since capital is fixed in the short run. While in the long run,

$$\Pi = PQ - (wL + rK) = (P \times F(K, L)) - (wL + rK)$$

And more generally, in a firm with N inputs, the short and long run profit function are,

$$\Pi = PQ - \sum_{i=1}^{n} c_i x_i - \sum_{j=N-n+1}^{N} c_j x_j = P \times F(x_1, x_2, \ldots, x_N) - \sum_{i=1}^{n} c_i x_i - \sum_{j=N-n+1}^{N} c_j x_j$$

and

$$\Pi = P \times F(x_1, x_2, ..., x_N) - \sum_{i=1}^{N} c_i x_i$$

Economists then solve this problem as a profit maximization problem. The question now is whether this is an accurate depiction of reality. Do firms really maximize profits?

Ignoring first non-profit organizations, and publicly owned firms, and focusing on private enterprises. Let's first consider the various manner in which a firm can be organized, 1. Sole Proprietorship, 2. Partnership, and 3. a Corporation. The key differential between all of them is the number of owners there are. Where the first type of firm has only one owner, and second has two or more. The last being typically listed and incorporated firms are owned by an even larger number of owners. Not all of them having the same rights in the operation of the firm, which is dependent on the number of shares of the firm the individual or group of individuals own. A share or stock in an incorporated firm tell you how much stake the individual has of the firms operation. But her liability is only for the amount of the value of the stake the individual has.

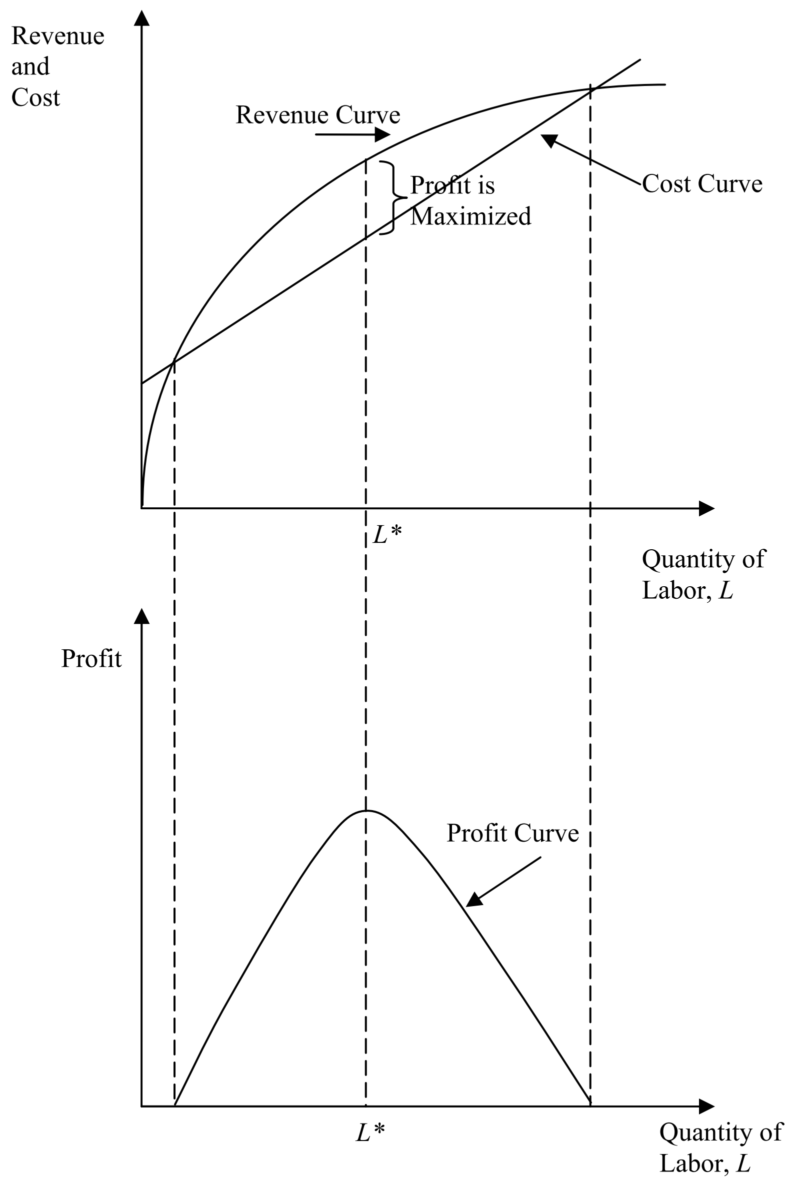
Let's examine the technology the firm could use. It is difficult to imagine that if there is a more efficient method of production that could reduce the cost of production, that the owner of the firm would choose not to, since the reduction in the cost of production means that the profit, or gains in producing the item rises, granted that we are assuming that the technology considered both yield products of the same quality. This then means that it is not unreasonable for us to think of firms as a profit maximizing entity.

Consider next public, and non-profit organizations. Now suppose we are thinking of the non-profit organization which offering a product or service for the common good of all. If for the product they provide or produce, if the operators had a choice between a cheaper more efficient way of producing the product, would they chose the less efficient and more efficient method. Noting that any increase in "profit" could go towards enhancing the product and service further, bringing more welfare to the populace it services. Hence, even in this case, it is still not unreasonable to think of these organizations being profit maximizing.

Profit Maximization
Short Run

In the short run, the technology, and capital is fixed, or more generally, there are some inputs that are fixed in the short run, and others that are not. Hence in considering maximization of inputs, a firm can only choose the optimal mix of variable inputs that yield the greatest profit. Consider the simplistic world where the firm only chooses labor and capital, where the latter is fixed in the short run. We will first examine the notion of profit maximization diagrammatically.

We have drawn the revenue curve of the firm as a concave curve below. Principally, this is because of the idea of diminishing marginal product. Focusing a single variable input model, where we can only vary labor demand, fixing the level of capital. The fixed cost with respect to the use of the capital tells us that the cost curve extend out from the vertical intercept. The slope of this cost curve is simply just the wage rate.



It should be clear from the above diagram that profit maximization is simply finding the greatest distance between the revenue function, and the cost curve. When the distance is greatest, it corresponds then to the maxima on the profit curve, or the highest point of the profit curve. Note that this is holding capital fixed in the short run, which in turn determines the optimal technology available to the firm. (It should also be clear why we need a concave shapes revenue curve. If the revenue curve is convex, there will not be a point where profit is maximized. In fact the optimal choice of labor is purely dependent on the size of the labor force the firm can hire.

How can we find such a point?
Consider the simple profit function we have in the short run above,

$$\Pi = PQ - \left(wL + r\overline{K}\right) = \left(P \times F\left(\overline{K}, L\right)\right) - \left(wL + r\overline{K}\right)$$

Then to find the optimal choice of labor input is equivalent to finding the peak of the profit function, which corresponds to the point where the slope of the profit function is zero. That is all we need to do is to differentiate the profit above with respect to quantity of labor L.

$$\frac{\partial \Pi}{\partial L} = \left(P \times \frac{\partial F\left(\overline{K}, L\right)}{\partial L}\right) - w = 0$$

$$\Rightarrow P \times MP_L = w$$

$$\Rightarrow VMP_L = w$$

What the condition says is that the firm should choose the optimal point such that the value of output that is created by the additional worker should just equate the wages paid to the worker. Logical? **Assuming the production function is of a Cobb-Douglas form, can you find out what the labor demand equation would be for such a firm?**

Is there any additional information to be gleaned from this condition? Recall comparative statics in consumer theory. We can examine this by varying the variables w, and P, that is wages and prices.

To examine the comparative statics, let's re-express the equilibrium condition as follows,

$$VMP_L = w$$

$$\Rightarrow MP_L = \frac{w}{P}$$

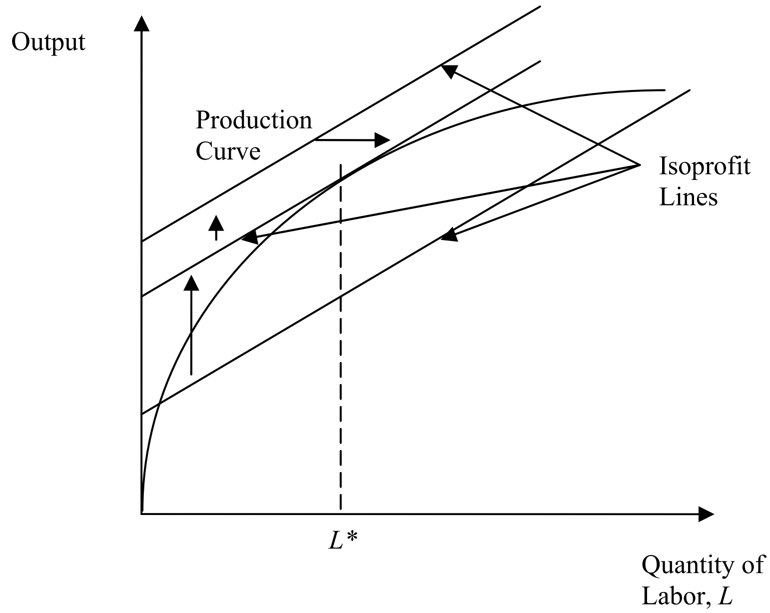
In this form, the point where the firm should choose for its output is where the marginal output of the additional worker is equal the ratio of wage and price of output. The question now is whether this is a intersection point or tangency of some sort? How would this approach affect how we think about the optimal choice? In fact, the solution above could similarly be obtained by maximizing the real profit, that is profit divided by the price of the good produced.

$$\frac{\Pi}{P} = \overline{\Pi} = Q - \left(\frac{w}{P}L + \frac{r}{P}\overline{K}\right)$$

Rearranging the latter part of the equation, we get

$$Q = \overline{\Pi} + \left(\frac{w}{P}L + \frac{r}{P}\overline{K}\right)$$

Graphing this relationship with Q being on the vertical axis, and L on the horizontal axis, we can we get what is called a isoprofit line, and may combine this line with the production curve. Note that since $\frac{w}{P}$ which is just the real wage paid for production of the product (This conception of a real wage is not the same as what you have in Macroeconomics, since the price is for the price of the good produce. Think of it as a wage-price ratio).This line is a positively slope line in this case since $\frac{w}{P}$ is positive. Diagrammatically, the problem is as follows,



Since the intercept is determined by the size of the profit (the greater the profit, the higher the isoprofit line), together with the equilibrium condition that we require, that the marginal product of labor be equal to the slope of the isoprofit line, profit maximization must be at the point where the isoprofit is just tangent to the production curve. If the isoprofit intersects the production curve, the firm can always do better by raising the isoprofit line and attaining a higher level of profit. If the isoprofit line is above the production curve, there is no feasible choice of labor input that can yield the desired profit.

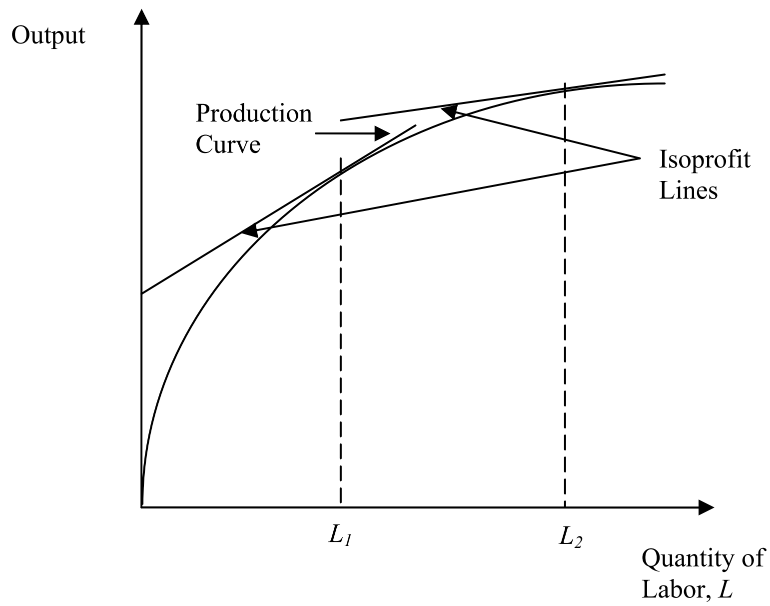
We can now examine how changes in prices and/or wages would affect the optimal choice of labor, or more generally any variable input. From the second representation of the equilibrium condition,

$$MP_L = \frac{w}{P}$$

it says that when the ratio of wage over price of product changes, so too does the optimal level of labor hired. To be precise,

1. When prices increase, the firms would produce at a lower level of marginal product of labor, which corresponds with a point on the production function with a lower slope compared to the original. (The result is reversed for a price decrease.) Intuitively, when the price increases, the revenues increase, which means the value of marginal product increase. This then means that there is more profit to be made from increasing output, and consequently necessitating an increase in hires.
2. When wage increase, the firms would produce at a higher level of marginal product of labor, which corresponds with a point with a steeper slope along the

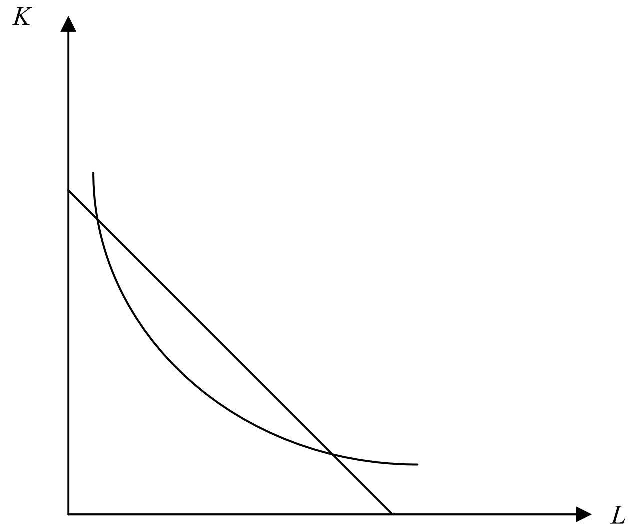
production curve, which in turn corresponds with a lower level of labor hired. (Again, the reverse is true). Intuitively, when price of output stays the same, the increase in wage implies a higher cost of production, which in turn reduces the profitability, necessitating a reduction in output, and hence the amount of labor hired.



What is the optimal choice of inputs in the Long Run?

In the long run, all inputs can be altered. When this occurs, how would equilibrium condition be altered? We can in fact make use of the ideas we used in Consumer Theory. If we restrict as above to considering two inputs, labor and capital, where the latter is fixed in the short run, it is possible to conceive of equilibrium as tangency between the isoquant of a firm against some cost function. If that is what thought, you are absolutely correct. That cost function in its simplest form has already been described above. We call the cost of a firm in this case here not the budget constraint, but the isocost line. Intuitively, the isoquant determines how much a firm can produce, or would like to produce to maximize their profit. The question next is what should be the optimal combination of inputs to achieve that level of profit.

Would it be an intersection? Well it cannot be, since if the isoquant is what the firm wants to attain, the firm can always produce more efficiently by producing at a lower cost level, described by a lower isocost line. Can the isocost not touch the isoquant at all? That cannot be, since that the firm is using too little inputs if they really want to produce the said quantity. That leaves us solely with tangency, where we have a unique choice where the isoquant just touches the isocost. This latter point means that the choice of quantity is feasible, and at the same time is the lowest possible cost the firm can produce the quantity at.



This idea is equivalent to maximizing the profit equation with respect to the best choice of *K* and *L*. Therefore differentiating the profit function,

$$\Pi = PQ - (wL + rK) = (P \times F(K, L)) - (wL + rK)$$

with respect to *K* and *L* we obtain these two conditions,

$$\left(P \times \frac{\partial F(K, L)}{\partial K}\right) = P \times MP_K = VMP_K = r$$

$$\left(P \times \frac{\partial F(K, L)}{\partial L}\right) = P \times MP_L = VMP_L = w$$

Recall from the above discussion of isoquants, we noted that the slope of the isoquant is just the ratio of the marginal product of labor over the marginal product of capital. This means that we can express the two conditions as one as follows,

$$\frac{VMP_L}{VMP_K} = \frac{P \times MP_L}{P \times MP_K} = \frac{MP_L}{MP_K} = \frac{w}{r} = -MRTS$$

Using the production function, and the above, we can then solve the optimal choice of labor and capital in terms of wages and rents. We will solve for a specific example during class.

The solution we find here is know as the factor demand function or curves.

Note

There is something interesting to be noted here. Let us express the optimal choice of labor and capital with a subscript *. Then the profit for the choice the firm made is

$$\Pi = (P \times F(K^*, L^*)) - (wL^* + rK^*)$$

Next suppose the firm has increasing returns to scale. What would happen to the profit of the firm? First note that increasing both labor and capital by say doubling the quantities of inputs based on what we know of returns to scale tells us that output would more than double. This then means that the revenue would more than double. Which means the firm could have double inputs and yet raised profits even further. How can that be if our solution is profit optimizing? That is a contradiction that our claim that the condition gives us the profit maximizing choices.

If you realize this, you would be correct. While partially. We neglected to note an important assumption when we set up the profit function the manner we did. What manner? Recall from your first year economics that a perfectly competitive firm is a **Price Taker**, meaning they cannot dictate the price they sell their goods at, and that the profit of a perfectly competitive firm is zero. If we realize this, what this means than when we double inputs (and everyone else in the perfectly competitive market doubles their inputs) is that they do in fact double output, but that would mean that they would have reduced prices, ensuring everyone earns zero profits.

This should give you a hint about how we should structure the profit function when we assume the firm can earn positive profits, and that they are price setters. (**Hint: Is the price a function of quantity produced?**)

Another question that you could ponder over is as follows: Can you prove that all it takes is two firms to have a perfectly competitive outcome of zero profits? Assuming both firms produce exactly the same good, and cannot differentiate themselves.