Advanced Macroeconomics, ECON 402
Optimal Control Theory

1 Introduction to Optimal Control Theory

With Calculus of Variations "in the bag", and having two essential versions of Growth Theory, we are now ready to examine another technique for solving Dynamic Optimization problems. The principle reason we need another method is due to the limitations to associated with *Calculus of Variations*:

1. Differentiable functions, and
2. Deals with Interior Solutions

Optimal Control Theory is a modern approach to the dynamic optimization without being constrained to Interior Solutions, nonetheless it still relies on differentiability. The approach differs from Calculus of Variations in that it uses *Control Variables* to optimize the functional. Once the *optimal path or value* of the control variables is found, the solution to the *State Variables*, or the optimal paths for the state variables are derived. We will denote as before state variables by $y(t)$, but now include control variables which will be denoted by $u(t)$. How do we decide which variable is a state and which is a control variable? A variable qualifies as a *control variable* if,

1. the variable is subject to the optimizer's choice, and
2. the variable will have an effect on the value of the state variable of interest.

For example, in creating universal laws to regular fishing, the stock of fishes in the seas is a state variable, and the control variable would be the rate at which fishes are fished out of the waters, which affects obviously the stock of fishes.

There are several interesting features to Optimal Control Problems:

1. The Optimal Path for the Control Variable can be piecewise continuous, so that it can have discrete jumps. Note however that the jump(s) cannot have $u(t) = \infty$.

2. The Optimal Path for the State Variable must be piecewise differentiable, so that it cannot have discrete jumps, although it can have sharp turning points which are not differentiable. Like the control variable $y(t)$ must be finite.

3. Each turn on by the State Variable along its path is associated with a discrete jump by the Control Variable.

4. A Optimal Control Problem can accept constraint on the values of the control variable, for example one which constrains $u(t)$ to be within a closed and compact set. This then allows for solutions at the corner.

The simplest Optimal Control Problem can be stated as,

$$\begin{aligned} \max V &= \int_0^T F(t, y, u)dt \qquad (1)\\ \text{subject to} \qquad \dot{y} &= f(t, y, u) \\ y(0) &= A \quad ,A \text{ is given} \\ y(T) & \quad \text{Free} \\ u(t) &\in \mathscr{U} \ \ \forall \ \ t \in [0, T] \end{aligned}$$

Note that to change the problem to a minimization problem, all one needs to do is to add a negative sign to the objective functional. There are several things you should note with the change in the statement of the problem,

1. The direction of the optimal path for the state variable, $y'(t)$ is now replaced by the control variable $u(t)$ in the objective functional. This highlights the fact that the role of the control variable has to be related to the state variable.

2. The relationship between the control variable and the state variable is described by the constraint $\dot{y} = f(t, y, u)$.

3. At the initial time, $t = 0$, and $y(0)$, so that $\dot{y}(0) = f(0, y(0), u(0))$, where $u(0)$ is chosen by us. In other words, the direction of the control variable at initial time is completely determined by our choice of $u(t)|_{t=0}$. Thus, it becomes clear that the direction that the state variable takes, $\dot{y}$ is determined by the control variable. Consequently, this equation is known as the *Equation of Motion* for the state variable.

4. Note that the constraint on the control variable $u(t) \in \mathscr{U}$ can be either a closed and compact set, or a open set, $\mathscr{U} = (-\infty, \infty)$. When the latter is the case, it obviates the use of the constraint, since there is essentially no constraint on the control variable.

5. When the equation of motion takes the form of $\dot{y} = u(t)$, then the problem reduces to our standard vertical terminal line problem in our Calculus of Variations discussion.

2 The Maximum Principle

The first order (necessary) condition in Optimal Control Theory is known as the *Maximum Principle*, which was named by L. S. Pontryagin. Firstly, to solve a Optimal Control problem, we have to change the constrained dynamic optimization problem into a unconstrained problem, and the consequent function is known as the *Hamiltonian function* denoted as H,

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda(t) f(t, y, u) \tag{2}$$

where $\lambda(t)$ is known as the *costate variable*. It's interpretation is not unlike the multiplier in Lagrange Multiplier method, in other words, you can think of it as a shadow price of the relationship between the state and control variable defined by $f(.)$.

Although, the norm in economics is to write the Hamiltonian in the above form, strictly speaking, there should be a constant multiplier as a coefficient to the objective functional integrand $F(.)$, since it should be verified that the integrand is not superfluous in the problem. Should the integrand be dispensable in the problem, then this constant term would be equal to zero. However, for all practical purposes in economics, this is hardly ever the case, and this constant is greater than zero, so that you can treat $\lambda(t)$ as the normalized value. In other words,

$$\begin{aligned} H'(t, y, u, \lambda) &= \lambda_0 F(t, y, u) + \lambda'(t) f(t, y, u) \\ \Rightarrow H(t, y, u, \lambda) &= F(t, y, u) + \frac{\lambda'(t)}{\lambda_0} f(t, y, u) \\ &= F(t, y, u) + \lambda(t) f(t, y, u) \end{aligned}$$

Read Chiang (1992) for a more in depth discussion.

The Maximum Principle is unlike the *Euler Equation* which is a single first order condition. The first order condition to the Hamiltonian, equation (2) is very much like the first order conditions of the Lagrangian, where they are obtained differentiating the Hamiltonian with respect to the control (if the Hamiltonian is differentiable with respect to the control variable), state and costate variables. We can write them as,

$$\max_u H(t,y,u,\lambda) \qquad \forall \;\; t \in [0,T] \tag{3}$$
$$\dot{y} = \frac{\partial H}{\partial \lambda} \qquad \text{Equation of Motion for } y \tag{4}$$
$$\dot{\lambda} = -\frac{\partial H}{\partial y} \qquad \text{Equation of Motion for } \lambda \tag{5}$$
$$\lambda(T) = 0 \qquad \text{Transversality Condition} \tag{6}$$

Notice that the first, first order condition with respect to the control variable is not stated as a derivative. This principally because, as note previously, Optimal Control Theory does allow for corner solution, which are points at which the Hamiltonian is not differentiable. Another way we can state the first condition of the Maximum Principle is,

$$H(t,y,u^*,\lambda) \;\; \geq \;\; H(t,y,u,\lambda) \;\; \forall \;\; t \in [0,T], \;\; \forall \;\; u^* \neq u, \;\; u, u^* \in \mathscr{U} \tag{7}$$

Note in addition the following

1. First note that for conditions described in equations (4) and (5), and are referred to as the *Hamiltonian System* or the *canonical system*,
2. Secondly, although equation (4) is a mere restatement of the relationship between the state and control variable, the equation of motion for λ is set such that $\dot{\lambda}$ equates with the negative of the derivative of the Hamiltonian function.
3. Finally, both the equation of the Hamiltonian system are first order differential equations, and there is no differential equation for the control variable.

2.1 A Simple Example

Consider the problem:

$$\begin{aligned} \max_u V &= \int_0^T -(1+u^2)^{1/2} dt \\ \text{subject to } \; \dot{y} &= u \\ y(0) &= A \;\; \text{A is given} \\ y(t) & \quad \text{Free, } \; \text{T is given} \end{aligned}$$

To solve this Dynamic optimization problem, we follow the following steps,

1. **Write down the Hamiltonian and first order condition with respect to the control variable** u:

   The Hamiltonian of the problem is,

   $$H = -(1+u^2)^{1/2} + \lambda u$$

   and since the integrand of the functional is differentiable, we can write the first order condition with respect to u as,

   $$\begin{aligned} \frac{\partial H}{\partial u} &= -(1+u^2)^{-1/2}u + \lambda = 0 \\ \Rightarrow (1+u^2) &= \frac{u^2}{\lambda^2} \\ u^2\left(1 - \frac{1}{\lambda^2}\right) &= -1 \\ u^2 &= \left(\frac{\lambda^2}{1-\lambda^2}\right)^{1/2} \end{aligned}$$

   You should verify that the Hamiltonian is concave in u. Next in light of the fact that the control variable is dependent on the costate variable, we would need to solve for the solution for λ.

2. **Solve for the optimal value of the costate variable** λ:

   Since the integrand is not dependent on the state variable, y

   $$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial y} = 0 \\ \Rightarrow \lambda &= k \end{aligned}$$

   where k is a constant. Since the transversality condition for this terminal line problem is $\lambda(T) = 0 = \lambda(t)$ for all $t \in [0, T]$, $\lambda^* t = 0$, which in turn implies that $u^*(t) = 0$ for $t \in [0, T]$.

3. **Solve for the Optimal Path of the state variable** y:

   Since $\dot{y} = u = 0$, this means that $y(t) = K$, where K is a constant. Since the initial state is $y(0) = A$, we have $K = A$, and $y^*(t) = A$, and the optimal path in this vertical terminal line problem is a horizontal straight line.

3 The Intuition Behind Optimal Control Theory

Since the proof, unlike the Calculus of Variations, is rather difficult, we will deal with the intuition behind Optimal Control Theory instead. We will make the following assumptions,

1. u is unconstrained, so that the solution will always be in the interior. In other words, $\mathscr{U}$ is an open set.

2. H is differentiable in all its arguments.

3. The initial point is given, but the terminal point is allowed to vary.

To restate the problem again,

$$\begin{aligned} \max V &= \int_0^T F(t,y,u)dt \qquad (8) \\ \text{subject to} \quad \dot{y} &= f(t,y,u) \\ y(0) &= y_0 \quad ,y_0 \text{ is given} \\ y(T) & \quad \text{Free} \\ u(t) &\in \mathscr{U} \ \ \forall \ \ t \in [0,T] \end{aligned}$$

1. First note that since the equation of motion will always be obeyed, $\dot{y} - f(t,y,u) = 0$ for all values of t, and $\lambda(t)(f(t,y,u) - \dot{y}) = 0$ for all values of t as well.

2. This in turn implies that,

$$\begin{aligned} \mathscr{V} &= \int_0^T F(t,y,u)dt + \int_0^T \lambda(t)(f(t,y,u) - \dot{y})dt \qquad (9) \\ &= \int_0^T \left(F(t,y,u) + \int_0^T \lambda(t)(f(t,y,u) - \dot{y}) \right) dt \qquad (10) \end{aligned}$$

3. Relating the above to the Hamiltonian, note next that,

$$\mathscr{V} = \int_0^T \left(H(t,y,u,\lambda) - \lambda(t)\dot{y} \right) dt \qquad (11)$$

4. Next integrate the second integral by part,

$$\begin{aligned} -\int_0^T \lambda(t)\dot{y}dt &= -\left(\lambda(t)y(t)\right)\big|_0^T + \int_0^T y(t)\dot{\lambda}dt \\ &= -\lambda(T)y_T + \lambda(0)y_0 + \int_0^T y(t)\dot{\lambda}dt \end{aligned}$$

5. The new objective functional is now,

$$\mathscr{V} = \int_0^T \left(H(t,y,u,\lambda) + y(t)\dot{\lambda}\right) dt - \lambda(T)y_T + \lambda(0)y_0$$

   Noting that the first integrand is a decision that spans the entire planning horizon, the second integrand is concerned with the terminal point, while the last integrand is concerned with the initial point. Also notice that the value of $\mathscr{V}$ dependent on the optimal paths for y, u, λ, and the values of T and y_T.

6. Recalling that λ is just a Lagrange multiplier, since $\dot{y} - f(t,y,u) = 0$ for all values of t, the path of λ is not important, so that without further complication, we can impose the constraint as a necessary condition,

$$\dot{y} = f(t,y,u) = \frac{\partial H}{\partial \lambda} \tag{12}$$

7. As in our discussion in Calculus of Variation, to understand the optimal path of the control variable, we can examine how an optimal control path differs from other possible paths, and thereby find out what distinguishes the optimal control path. To that extent, let

$$u(t) = u^*(t) + \epsilon p(t) \tag{13}$$

   where as before ϵ is just a perturbation constant, and $p(t)$ is a perturbation function.

8. This paths corresponds with a state path determined by the equation of motion, so that

$$y(t) = y^*(t) + \epsilon q(t) \tag{14}$$

9. Further, since T and y_T are variable,

$$\begin{aligned} T &= T^* + \epsilon\Delta T && (15) \\ \Rightarrow \frac{dT}{d\epsilon} &= \Delta T && (16) \\ y^T &= y_T^* + \epsilon\Delta y_T && (17) \\ \Rightarrow \frac{dy_T}{d\epsilon} &= \Delta y_T && (18) \end{aligned}$$

10. As before, noting that y and u are all functions of ϵ, we can think of $\mathscr{V}$ as being functions of ϵ. Then rewriting $\mathscr{V}$,

$$\mathscr{V} = \int_0^{T(\epsilon)} \left(H(t, y^* + \epsilon q(t), u^* + \epsilon p(t), \lambda) + (y^* + \epsilon q(t))\dot{\lambda} \right) dt - \lambda(T) y_T + \lambda(0) y_0$$

11. Differentiating the above with respect to ϵ to find the optimal value of ϵ to maximize $\mathscr{V}$, we obtain the following first order condition,

$$\begin{aligned}
\frac{\partial \mathscr{V}}{\partial \epsilon} &= \int_0^{T(\epsilon)} \left[\frac{\partial H}{\partial y} q(t) + \frac{\partial H}{\partial u} p(t) + \dot{\lambda} q(t) \right] dt + \left[H + \dot{\lambda} y \right]_{t=T} \Delta T \\
&\quad - \left[\lambda(T) \Delta y_T + y_T \frac{\partial \lambda(T)}{\partial T} \Delta T \right] \\
&= \int_0^{T(\epsilon)} \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda} \right) q(t) + \frac{\partial H}{\partial u} p(t) \right] dt + [H]_{t=T} \Delta T - \lambda(T) \Delta y_T = 0
\end{aligned}$$

Since each of the three components to the above solution has individual arbitrary components, each of them has to be independent equal to zero on the optimal path.

(a) Firstly, since $p(t)$ is arbitrary, then $\frac{\partial H}{\partial y} + \dot{\lambda} = 0$, which implies,

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \tag{19}$$

as required, and we have the equation of motion for the costate variable.

(b) Secondly, since $q(t)$ is arbitrary,

$$\frac{\partial H}{\partial u} = 0 \tag{20}$$

as required, which is that u has to maximize the Hamiltonian. Remember that we have this condition as a derivative because we had assume, to start, that the set of $\mathscr{U}$ is a open set.

(c) Since in the simplest problem T is fixed, $\Delta T = 0$, and we have no concerns there. However, for other problems, such as the Horizontal Terminal Line problem, ΔT is arbitrary, which would require that

$$H_{t=T} = 0 \tag{21}$$

which is the transversality condition for that problem.

(d) Finally, since Δy_T is arbitrary, we have

$$\lambda(T) \quad = \quad 0 \tag{22}$$

which explains the transversality condition in the simplest Optimal Control Problem.

12. In the case for the Terminal Curve problem, we need a definition or function which defines the behavior of the terminal point, such as $y_T = \phi(T)$. This then tells us that the transversality condition should be as follows,

$$\begin{aligned} [H]_{t=T} \Delta T - \lambda(T)\Delta y_T &= [H]_{t=T} \Delta T - \lambda(T)\phi'(T)\Delta T \\ &= [H - \lambda(t)\phi'(t)]_{t=T} \Delta T \end{aligned}$$

and since ΔT is arbitrary,

$$[H - \lambda(t)\phi'(t)]_{t=T} = 0 \tag{23}$$

which is the transversality condition for the terminal curve problem.

13. Finally, recall that in deriving the Euler Equation for the Elementary Problem in Calculus of Variation, we used the idea of a perturbation factor $p(t)$ to find the difference between all other paths for the state variable as compared against that of the optimal path. In the case of optimal control theory, this problem is complicated since conceptually, in deriving the solution, we are seeking to find the optimal means of "controlling" the control variable so as to influence and ultimately achieve an optimal path for the state variable. When the terminal point is fixed, any perturbation of the state variable is defined by the equation of motion for the state variable, which consequently implies the control variable is "not in control". Nonetheless, this is not a substantial problem since all it entails is that we change the transversality condition to

$$y(t) = y_T \tag{24}$$

where both T and y_T are given.

4 Current-Value Hamiltonian

In economic problems, as you well know by now, the objective integrand typically involves a discount factor, $e^{-\rho t}$. So that you can think of the objective integrand being of the form,

$$F(t, y, u) = G(t, y, u)e^{-\rho t} \tag{25}$$

This then changes the Hamiltonian function to,

$$H = G(t, y, u)e^{-\rho t} + \lambda f(t, y, u) \tag{26}$$

As you may well see, this adds an additional layer of complication, which of course you can work through. However, it turns out that you can simplify the Hamiltonian that so that it does not contain the the discount factor. The resultant Hamiltonian is known as the Current-Value Hamiltonian. As you may observe, this aim could be achieved by normalizing the Hamiltonian function by dividing it throughout by $e^{\rho t}$. Then all we need to do is to redefine the Lagrange Multiplier. Note then that after the normalization, we would have the Lagrange Multiplier in the Hamiltonian as $e^{\rho t}\lambda(t)$. Let us then denote it as $\gamma(t) = e^{\rho t}\lambda(t)$. Similarly, we would have to change the notation for the Hamiltonian, since it is now He^{pt}, which we will denote as H_c.

This then necessitates an examination of the Maximum Principle for this new Hamiltonian.

1. Since the discount factor is not dependent on the control variable u. The condition that u must maximize H_c for all values of t stands.

2. The equation of the state variable y should now be,

$$\dot{y} = \frac{\partial H_c}{\partial \gamma} \tag{27}$$

3. To obtain the equation of motion for the costate variable, first note that $\gamma = e^{\rho t}\lambda$ or $\lambda = e^{-\rho t}\gamma$. Next note that the previous condition was $\dot{\lambda} = \frac{\partial H}{\partial y}$. Then,

$$\dot{\lambda} = -\rho e^{-\rho t}\gamma + e^{-\rho t}\dot{\gamma} \tag{28}$$

$$\frac{\partial H}{\partial y} = \frac{\partial H_c}{\partial y}e^{-\rho t} \tag{29}$$

Putting the two equations together,

$$-\rho e^{-\rho t}\gamma + e^{-\rho t}\dot{\gamma} = -\frac{\partial H_c}{\partial y}e^{-\rho t} \tag{30}$$

$$\Rightarrow \dot{\gamma} = -\frac{\partial H_c}{\partial y} + \rho\gamma \tag{31}$$

4. Finally, the transversality condition $\lambda(T) = 0$ should now be $\gamma(T)e^{-\rho t} = 0$, and for $[H]_{t=T} = 0$ is now $[H_c e^{-\rho t}]_{t=T} = 0$, or $[H_c]_{t=T}e^{-\rho T}$.

We will not be discussing, as in our discussion of the Calculus of Variations, the necessary conditions for optimal control. Nonetheless, should you be interested, you can refer to Chiang (1992).

5 Infinite Horizon Problems

Just like Calculus of Variations has problems with handling infinite horizon problems, so too does Optimal Control Theory, and in fact L. S. Pontryagin paid cursory notice of the problem. Nonetheless, under certain conditions we would still be able to use Optimal Control Theory. Particularly, when $\lim_{t\to\infty} y(t) = y_\infty$, in other words, the state variable converges, since in that instance it is equivalent to the problem of a horizontal line problem, or that the Optimal Control problem converges towards such a problem.

What are the implication on the transversality conditions then when we are able to use Optimal Control Theory. Consider the equation we obtained earlier,

$$\frac{\partial \mathscr{V}}{\partial \epsilon} = \int_0^{T(\epsilon)} \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right)q(t) + \frac{\partial H}{\partial u}p(t)\right]dt + [H]_{t=T}\Delta T - \lambda(T)\Delta y_T = 0$$

As in our discussion in Calculus of Variations, what we need in the case of the infinite horizon problem is the following,

$$\frac{\partial \mathscr{V}}{\partial \epsilon} = \int_0^{\infty} \left[\left(\frac{\partial H}{\partial y} + \dot{\lambda}\right)q(t) + \frac{\partial H}{\partial u}p(t)\right]dt + \lim_{t\to\infty}[H]\Delta T - \lim_{t\to\infty}\lambda(t)\Delta y_T = 0$$

and since both ΔT and Δy_T are arbitrary, we know we then need,

$$\lim_{t\to\infty}[H] = 0 \tag{32}$$

$$\lim_{t\to\infty}\lambda(t) = 0 \tag{33}$$

References

Chiang, A. C. (1992): *Elements of Dynamic Optimization*. McGraw-Hill, New York: New York.