Mathematical Economics (ECON 471)
Lecture 5
Homogeneous & Homothetic Functions

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We will round out our discussion on constrained optimization with a class of functions that occurs rather frequently in Economics, ***Homogeneous Function***.

Definition 1 *For any scalar α, a real valued function $f(\mathbf{x})$, where $\mathbf{x}$ is a $n \times 1$ vector of variables, is homogeneous of degree α if*

$$f(t\mathbf{x}) = t^{\alpha} f(\mathbf{x})$$

for all $t > 0$

It should now become obvious the our profit and cost functions derived from production functions, and demand functions derived from utility functions are all homogeneous functions. Using these functions offers us ease of interpretation of key economic ideas. Consider the following idea related to production functions, the **returns to scale**. Let $f(\mathbf{x})$ be the production function. Then if it were homogeneous of degree $\alpha = 1$, it would be associated with **constant returns to scale**. If $\alpha > 1$, then the firm would be experiencing **increasing returns to scale**. On the other hand, if $\alpha < 1$, the firm would be experiencing **decreasing returns to scale**. It should become clear why the **Cobb-Douglas** function is so popular in undergraduate economics.

Some of the key properties of a homogeneous function are as follows,

1. For a twice differentiable homogeneous function $f(\mathbf{x})$ of degree α, the derivative is

homogeneous of degree $\alpha - 1$. This can be easily proved,

$$\begin{aligned} f(t\mathbf{x}) &= t^{\alpha} f(\mathbf{x}) \\ \Rightarrow t\frac{\partial f(t\mathbf{x})}{\partial t x_i} &= t^{\alpha}\frac{\partial f(\mathbf{x})}{\partial x_i} \\ \Rightarrow \frac{\partial f(t\mathbf{x})}{\partial t x_i} &= t^{\alpha-1}\frac{\partial f(\mathbf{x})}{\partial x_i} \end{aligned}$$

2. Tangent planes to the level sets[1] of the function $f(.)$ along each ray extending from the origin yields have a constant gradient. This may be proved easily as well. First recall that the slope of the homogeneous function (such as an isoquant and indifference curve) is just the ration of the partials derivatives of the function with respect to the respective variables depicted on a graph. That is,

$$\begin{aligned} \frac{f_{x_i}(t\mathbf{x})}{f_{x_j}(t\mathbf{x})} &= \frac{t^{\alpha} f_{x_i}(\mathbf{x})}{t^{\alpha} f_{x_j}(\mathbf{x})} \\ &= \frac{f_{x_i}(\mathbf{x})}{f_{x_j}(\mathbf{x})} \end{aligned}$$

3. The implications of the above on the utility and production function are then as follows:

   (a) If $U(\mathbf{x})$ is a homogeneous utility function of degree α, then

      i. the MRS is constant along rays extending from the origin,
      ii. the income expansion paths are rays extending from the origin,
      iii. the demand derived from utility maximization is a linear function of income,
      iv. the income elasticity of demand is 1 for the demand.

   (b) If $F(\mathbf{x})$ is a homogeneous production function of degree α, then

      i. the MRTS is constant along rays extending from the origin,
      ii. the corresponding cost function derived is homogeneous of degree $1/\alpha$.

4. Euler's Theorem can likewise be derived. The theorem says that for a homogeneous function $f(\mathbf{x})$ of degree α, then for all $\mathbf{x}$

$$x_1\frac{\partial f(\mathbf{x})}{\partial x_1} + \cdots + x_n\frac{\partial f(\mathbf{x})}{\partial x_n} = \alpha f(\mathbf{x})$$

[1] The level sets of a homogeneous function are radial expansions and contractions of each other, much like you isoquants, and indifference curves. But homogeneous functions are in a sense symmetric.

To proof this, first note that for a homogeneous function of degree α,

$$\begin{aligned}\frac{df(t\mathbf{x})}{dt} &= \frac{\partial f(t\mathbf{x})}{\partial tx_1}x_1 + \cdots + \frac{\partial f(t\mathbf{x})}{\partial tx_n}x_n \\ \frac{dt^\alpha f(\mathbf{x})}{dt} &= \alpha t^{\alpha-1} f(\mathbf{x})\end{aligned}$$

Setting $t = 1$, and the theorem would follow. Note further that the converse is true of Euler's Theorem.

Since a homogeneous function has such great features, it would be perfect if we can "create" them in some sense, and we can. Before we examine this, we need to define what a cone C in $\mathbf{R}^n$ is. A set $C \subset \mathbf{R}^n$ is a cone (with vertex at zero) if $x \in C$ implies $\lambda x \in C$ for all positive scalar λ. Then for a real valued function $f(\mathbf{x})$ in $C \subset \mathbf{R}^n$, we can define a new function $F(.)$ of $n+1$ variables by,

$$F(\mathbf{x}, y) = y^\beta f(\mathbf{x}.1/y)$$

where β is an integer, so that $F(.)$ is a homogeneous function of degree β on the cone $C \times \mathbf{R}_+ \subset \mathbf{R}^{n+1}$. This means that you can think of $f(.)$ as a restriction of $F(.)$ to a n dimensional subset of $\mathbf{R}^{n+1}$. To proof this, note that for $t \in \mathbf{R}_+$ and $(\mathbf{x}, y) \in C \times \mathbf{R}_+$,

$$\begin{aligned}F(t\mathbf{x}, ty) &= (ty)^\beta f\left(t\mathbf{x}.\frac{1}{ty}\right) \\ &= (ty)^\beta f\left(\mathbf{x}.\frac{1}{y}\right) \\ &= t^\beta y^\beta f\left(\mathbf{x}.\frac{1}{y}\right) \\ &= t^\beta F(\mathbf{x}, y)\end{aligned}$$

The really neat thing is that the converse is true. That is for a real valued function $F(\mathbf{x}, y)$ in $C \times \mathbf{R}_+$, where $C \subset \mathbf{R}^n$, that is homogeneous of degree β, and

$$F(\mathbf{x}, 1) = f(\mathbf{x}) \;\; \forall \;\; \mathbf{x} \in C$$

then $F(\mathbf{x}, y) = y^\beta f\left(\mathbf{x}.\frac{1}{y}\right)$ for all $(\mathbf{x}, z) \in C \times \mathbf{R}_+$. The proof begins with us noting the since $F(.)$ is homogeneous of degree β,

$$\begin{aligned}F(\mathbf{x}, y) &= F\left(y.\left(\mathbf{x}.\frac{1}{y}, 1\right)\right) \\ &= y^\beta F\left(\mathbf{x}.\frac{1}{y}, 1\right) \\ &= y^\beta f\left(\mathbf{x}.\frac{1}{y}\right)\end{aligned}$$

Then this two theorems in essence allows us to "normalize" a function so that even if it began as rather untidy, it will be neater due to the homogenizing operation.

However, there is a caveat. Whenever we perform any transformations we have to ensure we are not destroying key features that are necessary for our analysis. Consider the utility function in consumer theory. Because the welfare idea in the function is not measurable, the ordering achieved for differing consumption bundles and its commiserating utility derived is *ordinal* in nature. Yet the functions we deal with, and homogeneity itself are *cardinal* properties. We will define these two concepts clearly first, and examine whether there is any cardinal content in the concept of homogeneity, and discuss subsequently a class of functions known as *homothetic functions*.

A concept is said to be ordinal if the function and mappings associated with it has little inherent meaning except to tell us that one value construed to be "better" than another, or that you can rank the mappings in terms of their relative locations. On the other hand, a concept is *cardinal* if the values yielded by the function has meaning. It is for the reason that utility functions has *ordinal* properties, that are not exactly compatible with the concept of homogeneity. This means that when transforming functions with *ordinal* properties, we have to ensure that the transformation does not "jumble up" the inherent order.

A **Monotonic Transformation** has those properties, and it is defined as follows.

Definition 2 *Let A be an interval on $\mathbf{R}^1$, that is $A \subset \mathbf{R}^1$. Then a transformation $f : A \longmapsto \mathbf{R}^1$ is a* ***monotonic transformation*** *of A if f is a strictly increasing function on A. In addition, if g is a real valued function of $\mathbf{x} \in \mathbf{R}^n$, then $f \circ g : \mathbf{x} \longmapsto f(g(\mathbf{x}))$, and we say that f is a monotonic transformation of g.*

We can likewise define ordinal and cardinal properties now in relation to monotonic transformations.

Definition 3 *A feature of a function is* ***ordinal*** *if every monotonic transformation of the function retains the original features. On the other hand,* ***cardinal*** *properties are not retained after a monotonic transformation since the inherent values of the original functions are now loss.*

This definitions are most important in consumer theory due to ordinal features of utility functions. However, although isoquants are similar diagrammatically to indifference maps associated with utility functions, because there is a cardinal quantity associated with each isoquant, the concerns of consumer theory does not cross over to producer theory. Put another way, a monotonic transformation of a production function is not innocuous, and will totally change the implications of profit maximization!

Given these difference in concept, and yet the neat properties associated with the homogeneous functions, we need to ask ourselves whether there is a class of functions that are homogeneous, and yet possesses all the cardinal properties so that we may use them in our consumer theory analysis. The good news is that there is, and they are known as **Homothetic** functions.

Definition 4 *A function $h : \mathbf{R}^n_+ \longmapsto \mathbf{R}_+$ is called **homothetic** if it is a **monotone transformation** of a homogeneous function. Put more formally, if there is a monotonic transformation such that $y \longmapsto f(y) \in \mathbf{R}_+$ and a homogeneous function $g : \mathbf{R}^n_+ \longmapsto \mathbf{R}_+$ so that $h(\mathbf{x}) = f(g(\mathbf{x}))$ for all $\mathbf{x}$ in the domain.*

It is the monotone transformation portion of the function that ensures that the new function retains the ordinal property. Your text book provides a prove as to the fact that a homotheticity is a ordinal concept. The prove relies on the fact that a monotone transformation of a monotone transformation remains a monotone transformation, and so the properties of homogeneity is retained, and it is a ordinal concept.

To get a more detailed understanding of homotheticity, we need the following additional definitions.

Definition 5 *If vectors $\mathbf{x}$ and $\mathbf{y}$, both in $\mathbf{R}^n$ are such that $\mathbf{x} \geq \mathbf{y}$, then $x_i \geq y_i$ for $i \in \{1, \ldots, n\}$. Alternatively, if $\mathbf{x} > \mathbf{y}$, then $x_i > y_i$ for $i \in \{1, \ldots, n\}$. A function $f : \mathbf{R}^n \longmapsto \mathbf{R}_+$ is then **monotone** if for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n_+$ we have $\mathbf{x} \geq \mathbf{y} \Rightarrow f(\mathbf{x}) \geq f(\mathbf{y})$. In addition, the function is strictly monotone if for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n_+$ we have $\mathbf{x} > \mathbf{y} \Rightarrow f(\mathbf{x}) > f(\mathbf{y})$*

This concepts go hand in hand in consumer theory where one of the principal assumptions we have is non-satiation, or "more is always better". We will now characterize several characteristics of homothetic functions.

Theorem 1 *For a strictly monotonic function $f : \mathbf{R}^n_+ \longmapsto \mathbf{R}_+$, it is homothetic if and only if for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n_+$.*

$$f(\mathbf{x}) \geq f(\mathbf{y}) \Longleftrightarrow f(t\mathbf{x}) \geq f(t\mathbf{y}) \;\; \forall \;\; t > 0 \tag{1}$$

The proof is instructive, and we will go through this carefully since we need to proof both sides of the condition.

Proof. To first show that if f satisfies condition (1) of theorem 1, it is homothetic, let $\mathbf{e}$ be a vector of ones that span the diagonal, Δ, in $\mathbf{R}^n$. This is necessary and useful as it will be against this ray from the origin that we can establish the distance from the origin, and eventually establish the cardinality. Next, let $h : \mathbf{R}_+ \longmapsto \mathbf{R}$ such that,

$$h(t) \quad = \quad f(t\mathbf{e})$$

for a scalar $t > 0$. Then the property that f is strictly increasing is translated onto the function h, and since h is increasing, it's inverse h^{-1} is likewise an increasing function. This then imply that $h \circ h^{-1} \circ f = f$. This is necessary since this would mean with the inverse of h, we would be able to know how far up the diagonal, Δ, a level set associated with any vector intersects the diagonal ray from the origin. This is where good definitions help in delivering your proof. To show the cardinal property, we need to show that the function f is homothetic. But we have defined previously that a function is homothetic if it is a monotone transformation of a homogeneous function. Since h is monotonic, it remains to show that $h^{-1} \circ f$ is homogeneous.

For any scalar a, the inverse of h, as noted prior, tells us how far up Δ the level set $h^{-1}(a)$ meets Δ. Therefore, for a vector $\mathbf{x}$, $h^{-1}(f(\mathbf{x}))$ tells us how far along the ray Δ from the origin does the level set through $\mathbf{x}$ intersects the ray Δ. Put simply, $t = h^{-1}(f(\mathbf{x}))$ is the solution to

$$f(\mathbf{x}) = f(t\mathbf{e}) \tag{2}$$

Let $\alpha > 0$ be a scalar, then by the condition (1),

$$f(\mathbf{x}) = f(t\mathbf{e}) \implies f(\alpha\mathbf{x}) = f(\alpha t\mathbf{e}) \tag{3}$$

But we know that,

$$\begin{aligned}
h(t) &= f(t\mathbf{e}) \\
\therefore\ h(\alpha t) &= f(\alpha t\mathbf{e}) \\
\Rightarrow\ \alpha t &= h^{-1}(f(\alpha t\mathbf{e})) \\
&= h^{-1}(f(\alpha x)) \\
\Rightarrow\ \alpha h^{-1}(f(t\mathbf{e})) &= \alpha h^{-1}(f(\mathbf{x})) \\
&= h^{-1} f((\alpha\mathbf{x}))
\end{aligned}$$

and $h^{-1} \circ f$ is homogeneous of degree 1. Therefore by using the definition, since h is monotonic, and $h^{-1} \circ f$ is homogeneous, then $h \circ h^{-1} \circ f = f$ is homothetic.

However, due to the statement of the theorem, the proof is incomplete. We have to show now that a homothetic function f will give rise to the condition (1). First suppose that f is homothetic so that $f = h_1 \circ k$ with h_1 being monotonically increasing, and k being homogeneous of degree one. Next let k be $h_2 \circ m$, where $h_2(z) = z^\beta$ and $m(\mathbf{x}) = q(\mathbf{x})^{1/\beta}$, where q is homogeneous of degree β. This is done to ensure that $k = h_2 \circ m$ is homogeneous of degree one. Also note that since both h_1 and h_2 are monotonically increasing, then $h_1 \circ h_2$ is monotonically increasing.

Suppose $f(\mathbf{x}) \geq f(\mathbf{y})$, and let $\alpha > 0$. Since h_1 and h_2 are monotonically increasing, it's inverse is likewise monotonically increasing. Therefore,

$$
\begin{aligned}
f(\mathbf{x}) &\geq f(\mathbf{y}) \\
\Rightarrow h_2^{-1}(h_1^{-1}(f(\mathbf{x}))) &\geq h_2^{-1}(h_1^{-1}(f(\mathbf{y}))) \\
\Rightarrow m(\mathbf{x}) &\geq m(\mathbf{y}) \\
\Rightarrow m(\alpha\mathbf{x}) = \alpha m(\mathbf{x}) &\geq \alpha m(\mathbf{y}) = m(\alpha\mathbf{y}) \quad \text{Since } k \text{ is homogeneous of degree 1} \\
\Rightarrow h_1(h_2(m(\alpha\mathbf{x}))) &\geq h_1(h_2(m(\alpha\mathbf{y}))) \\
\Rightarrow f(\alpha\mathbf{x}) &\geq f(\alpha\mathbf{y})
\end{aligned}
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Recall that the gradient of the level sets along the rays from the origin are all the same. This is an ordinal concept, and formally, we can state it as follows.

Theorem 2 *For a homothetic function f on $\mathbf{R}^n$, the gradient of the tangent planes to the level sets of f are constant along the rays from the origin. That is,*

$$
\frac{\frac{\partial f(t\mathbf{x})}{\partial x_i}}{\frac{\partial f(t\mathbf{x})}{\partial x_j}} = \frac{\frac{\partial f(\mathbf{x})}{\partial x_i}}{\frac{\partial f(\mathbf{x})}{\partial x_j}}
$$

for all $t > 0$.

Note that the converse of the theorem is true.