

# Physics 201: Notes on Experimental Uncertainties

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## Motivation

Error analysis is a crucial part of experimental physics and a requisite part of the course material. The identification of sources of error is critical to the conception and subsequent redesign of experiments. *However* the way it is presented to students is usually as a disjointed set of rules that have nothing but exceptions. Even after you have done all of the work you end up with a result and say “our result is...” 1) within error 2) not within error or 3) close to the accepted result and then trot out a line about “human error” or “measurement error”. Even for an undergraduate experiment a definite conclusion about the truth or untruth of physical laws is important. Proper error analysis is key to really answer your hypothesis. Error analysis is part statistics but to a large degree involves intelligent, frank *estimations* based on your experimental design, equipment, and results. From an instructor’s point of view, a student who understands error analysis understands the experiment, more so than a student who has a result that is close to the accepted value. From a real world point of view the concepts of tolerances, stability, and reproducibility are very important. From the other, real world point of view most people in the world have a very, very poor grasp of experimental uncertainties. (I have a couple of examples I have seen in the media over the past couple of years.) This can lead to significant misunderstandings between scientists and the general population e.g. global warming.

## What not to do

To be clear *error* or *uncertainty* is not a mistake but it is an *estimate* of the reliability of the result. Another common fallacy about error is that it is

$$\frac{(\text{your answer}) - (\text{right answer})}{(\text{right answer})} \times 100\%. \quad (1)$$

## Wrong!

1. For one thing there usually aren’t *right* answers. “Commonly accepted” is more correct (even if sounds more politically correct rather than physically correct). But the distinction is important. If we measure  $g$ , the gravitational acceleration, in the 3rd year lab it is not a value you can look up to arbitrary precision and say “that’s the right answer”. It varies in a measureable way from place to place in way that depends on latitude, altitude, and local geologic formation. Also tied into this is the acceleration due to the earth’s rotation.
2. Even with this proviso the above quantity indicates only how far away you are from a commonly accepted result. It may turn out that the result of equation (1) agrees with your error estimate but that doesn’t say much about your assessment of your experimental setup. And what would you do if there was no accepted value? Or if you wanted to claim there was

a reason for taking your value over the accepted value? Or if you wanted to challenge an existing theory? Or if you wanted to get a good mark?

3. That said once you have your quantity and your error estimate comparing them to the commonly accepted value is important (see below). I am just saying it isn't the end of the story.

### ... so what *is* the error estimate?

There are differences in the definition but the most common is that it is meant to represent a statistical uncertainty of one standard deviation. Suppose we make some measurement of  $g$  and we claim an error of  $\Delta g$ . I would expect that the actual value of  $g_0$  (or a more accurate measurement made with a smaller  $\Delta g$ ) would be within  $\Delta g$  of  $g$  68% of the time and within  $3\Delta g$  99% of the time. (This is the meaning of "within error".) Repeated measurements of the same quantity with the same apparatus under similar conditions should yield data that follow this distribution. Imagine a histogram of numerous results obtained by groups doing the same kind of experiment. Under these assumed conditions it would look like a Gaussian with a half width at half max of  $\Delta g$

### ...assuming

that the error estimates you have built in are random. e.g. measurements made with a collection of rulers that are both "too short" and "too long" as opposed to one ruler that is 5% too short.

### Accuracy versus precision

I assume you all know a bit about this. These terms are often used interchangeably and most of the time that is fine. However when discussing errors think of precision as more and more digits. (ruler to Vernier caliper to micrometer) and accuracy as closeness to the real value. So your quoted error estimate is a reflection of the accuracy but certainly is limited by precision. A stopwatch can be very precise (0.01 s) but can be inaccurate because of your ability to see the event being timed and push the button. A photogate can remove this lack of accuracy but then may be limited by how well the stopwatch runs according to accepted standards (good to know for the Olympics, assuming the wind isn't blowing too hard. Seriously.) Lack of precision and lack of accuracy are both included in the error.

### Types of errors

I won't always expect a deliniation of every error into one of these types but you may want to consider how your measurements fit into these categories (especially the dominant error).

1. Reading: linked to the digits of precision and ability to read the instrument. Can be reduced with more precise equipment if this is the limiting factor. Cannot be reduced by repeating the same measurement over and over. Classic examples: trying to read a ruler between the marks, output of digital multimeter, reading peak to peak voltage or period from an oscilloscope (although there are good and bad ways to do this). Assuming everything else is okay (calibration, a steady reading) a reading error estimate is usually half (digital) or a fifth (analog) the most significant digit.
2. Random: although you feel there is some correct value the measurement obtained fluctuates above and below this value. If your assumption of a correct value is sound the effect can

be reduced by repeated measurements. Classic examples: reaction time with a stop watch, estimating how a beam lines up with bar

3. Systematic: reflects a limitation beyond measuring and random errors and I will consider two types: 1) Contributions from the experimental equipment that results from poor or limited calibration. Classic example: not zeroing an instrument, making measurements at the high end of the scale where the instrument is no longer linear. These sources of error may be correctable if your instrument is stable. You might think of these as mistakes. 2) Experimental setup and the theory and assumptions that goes into the experiment (usually preceded by the statement “we will ignore”. Classic examples: air resistance, rigid rods, perfect capacitors.) One of the best ways to spot these errors is to look for deviations away from your theory. An improved theory or experiment may help.

Any or all of these can contribute to make an experiment impossible or impractical.

## Error propagation

These are the boring rules. You start with error *estimates* ( $\Delta a, \Delta b, \Delta c$ ) of the observed quantities ( $a, b, c$ ) and work from there to find the error *estimate*  $\Delta y$  in the calculated quantity  $y$ . The rules apply to non-correlated errors only but they can be applied sequentially.

1. Addition or subtraction: If  $y = a \pm b \pm c$  then

$$\Delta y = \sqrt{(\Delta a)^2 + (\Delta b)^2 + (\Delta c)^2} \quad (2)$$

Extends to however many quantities you need to add or subtract (errors add in quadrature).

2. Multiplication or division: If  $y = \frac{ab}{c}$  then

$$\frac{\Delta y}{y} = \sqrt{\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2} \quad (3)$$

Extends to however many quantities you need to divide or multiply. (Percentage errors add in quadrature.) Corollary: If  $y = a/N$  where  $N$  is a constant without uncertainty then  $\Delta y = y\Delta a/a = \Delta a/N$ . Same idea for multiplication by a constant. (The percentage errors are the same; the error is affected by the constant in the same way as the measured quantity.)

3. Repeated measurements and averaging or the error in the mean: A corollary of the first two rules. If you make  $N$  measurements of  $x$  each with random error  $\Delta x$  and the total is  $x_T$  and the average is  $\bar{x} = x_T/N$  then the error in the average

$$\Delta(\bar{x}) = \Delta\left(\frac{x_T}{N}\right) = \frac{\Delta x_T}{N} = \frac{\sqrt{N(\Delta x)^2}}{N} = \frac{\Delta x}{\sqrt{N}} \quad (4)$$

(divide by square root of the number of trials). This formula allows you to reduce random errors to a point before being limited by systematic and reading errors.

4. General: If  $y = f(x)$  where  $f$  is some function like cosine, exponential, log, power, etc. then

$$\Delta y = \left| \frac{\partial f}{\partial x} \right| \Delta x \quad (5)$$

You calculate the derivative at  $x$ . This assumes the errors are fairly small and the derivatives are well behaved over that range. I'll let you work out these formulas. Signs don't make a difference, hence the absolute value. Remember you need trig functions in radians for this to work in the simple way.

5. Really stuck?: You have calculated  $y$  for some value of  $x$ , calculate  $y$  again for  $x + \Delta x$ . The absolute value of the difference in  $y$  is your error.
6. Dominant: If you have made it this far and can't believe the amount of work you have to do just to get an error bar on some graph here are a few ideas:
  - (a) **There is usually one dominant error. Identify it!** Since the formulas for combining errors is in quadrature most of the non-dominant error sources can be discarded for your error estimate.
  - (b) If you use spreadsheets or something else that can calculate formulas just put the error formula in with it. (Now go back and look at the above rule.)
  - (c) For error bars on graphs just the first and last point is usually sufficient if you are doing calculations by hand.

### Error in the error and sig figs

As said repeatedly error is an estimate and at this level of analysis it doesn't make sense to quote it to more than two sig figs. And I would say only use two sig figs if the first digit is 1. (Keep 0.18 but round 0.23 to 0.2). The rules for sig figs you remember from high school are kind of very rough method for error propagation. Your finally quoted answer should go to the same decimal place as your quoted error estimate.

### Data analysis

Most plots from which we wish to derive quantities are linear or can be made linear. **Do not analyse your data by repeatedly applying and solving the formula to obtain the desired quantity. You will lose a lot of points if you do this.** The power of a plot is to see predictability and the region of validity of a physical law. Also as mentioned above you may be able to see systematic deviations from your expected behaviour. Most derived quantities are related to the slope. Estimates for the error in the slope can be made by drawing lines with maximum possible slope and minimum possible slope that is consistent with the data (its error bars and scatter). Be on the look-out for points that are more than 3 error bars off the best fit line. Maybe that point should not be included in your data fitting (check it, did you make a mistake?) If your error bars are too small compared to the range of  $y$  consider the high accuracy trick. Beware that the standard error quoted by most computer fitting programs is far smaller than a reasonable error estimate and doesn't take error bars or statistical weighting into account.

### High accuracy trick

The *high accuracy trick* is something I learned as a lab demonstrator at the University of Toronto. You measure a bunch of  $(x, y)$  points and put them on a plot (either a computer or by hand). Suppose that  $y$  values go over a range of 0 to 10. Now further suppose that your error bars in the  $y$  direction are 0.05 (almost the size of the "symbol" you use in the plot). You fit a straight line and it lies right on top of all of the points. A good example of this would be verifying Ohm's law using

a resistor and a digital multimeter. How do you get a max-min slope out of this since essentially they all lie right on top of each other? What you want to do is magnify the region around each point and see just how close your dead-on fit is compared to the error bar. You can't just expand the  $y$ -scale. What you can do is *transform* the  $y$ -values. Let the new  $y'$  values be

$$y' = y - mx \tag{6}$$

where  $m$  is an estimate of the slope or possibly slightly less than the fitted-slope.  $y'$  should have much smaller range than  $y$  (in fact if you really had  $y = mx + b$  then  $y'$  vs  $x$  would be a constant function equal to  $b$ ). Since the range is much smaller the error bars should look relatively bigger. Now do max-min slopes and estimate the uncertainty in your slope. At this level you may also see clear deviations from linear behaviour.