Midterm Quiz: Modern Physics 201
February 15, 2011
Solutions

Formula sheet provided. Total 32 points. Individual values follow each question. Usual conventions for frames S and S' apply. **I added 5 points to everyone's score because of a misleading question in 4(c).**

1. A particle is moving at $u_x = 0.9c$ in the lab frame. It has a rest mass of $0.8\ \mathrm{MeV}/c^2$.

   (a) Calculate its energy and momentum and give your answers in MeV-style units. (2)
   Answer: Use formulas 16 and 17. Both of them contain the $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 2.29$ factor ($u = u_x$ in this case). So $p = p_x = \gamma_u m u_x = (2.29)(0.8\ \mathrm{MeV}/c^2)(0.9\ c) = 1.65\ \mathrm{MeV}/c$ and $E = \gamma_u mc^2 = (2.29)(0.8\ \mathrm{MeV}/c^2)c^2 = 1.83\ \mathrm{MeV}$.

   (b) Another particle is coming from the right with $E = 2$ MeV and $p_x = -2\ \mathrm{MeV}/c$. What is its rest mass? What is its velocity? (2)
   Answer: Rearrange formula 18 to solve for mc^2 and obviously you end up with $m = 0\ \mathrm{MeV}/c^2$ To solve for the speed you can either just state that for a massless particle that $u = c$ (or $-c$ if you say it is moving from the right) as we discussed in class. Or divide formula 17 by formula 16 to give $u_x/c = \frac{E}{p_x c}$ and then you can see that $u_x = -c$. The m's cancel so it is okay to use the ratio of formulas for $m = 0$.

   (c) Calculate the rest mass of the two particles together as a system. (2)
   Answer: The key to this problem is to remember that you need to add the energies and momenta of the components *then* use formula 18 to solve for the mass. The momenta add as components (so "left" has the opposite sign from "right"). Using the subscript T for "total"

$$E_T = 1.83\ \mathrm{MeV} + 2\ \mathrm{MeV} = 3.83\ \mathrm{MeV} \quad (1)$$

$$(p_x)_T = 1.65\ \mathrm{MeV}/c - 2\ \mathrm{MeV}/c = -0.35\ \mathrm{MeV}/c \quad (2)$$

$$m_T = \frac{\sqrt{E_T^2 - p_T^2 c^2}}{c^2} = \frac{\sqrt{E_T^2 - (p_x)_T^2 c^2}}{c^2} = \frac{\sqrt{(3.83)^2 - (0.35)^2}}{c^2} = 3.81\ \mathrm{MeV}/c^2 \quad (3)$$

   (d) An observer in another frame moving at $v = 0.7c$ relative to the first observes the same system. Knowing what you know about 4-vectors and 4-scalars what *must* they calculate for the rest mass of the system? Give an example of a 4-vector by stating its components. (2)
   Answer: The rest mass of the system is a Lorentz invariant or a 4-scalar since it is composed of a "dot product" of the energy-momentum 4-vector with itself. A 4-scalar is unchanged by a Lorentz transform and the other observer must calculate the same value.
   The energy-momentum 4-vector is $(p^0, p^1, p^2, p^3) = (E/c, p_x, p_y, p_z)$. The position 4-vector is (ct, x, y, z). (Note: strictly speaking these are *contravariant* 4-vectors and the *covariant* 4-vectors have negative signs on components 1 through 3. The 4-scalar is composed of the "dot product" of covariant and contravariant 4-vectors.)

2. What is *relativity*? What types of frames are involved when we talk about "special" relativity? In *this* context explain why the Michelson-Morley experiment had a null result (i.e. no ether was detected). (3)

   Answer: Most of you knew that it had to do with observations or coordinates in different reference frames. But formally, *relativity*, as part of physics, *describes how different observers state the laws of physics.* The different observers set up convenient frames of reference for themselves with their own coordinates. A large part of relativity is studying how those coordinates transform between the different reference frames, and then you can study how different physical properties (such as time intervals, momentum, and energy) are changed. The *hope* of relativity is that even though everyone disagrees on the specific quantities that eventually everyone can agree on the form of the physical laws *once* you get the correct quantities in the laws. e.g. "$E_K = \frac{1}{2}mu^2$ and E_K is conserved" cannot be a relativistically correct physical statement because in this form E_K transforms in a different way for different observers. *Special relativity* is a terrible juxtaposition since the entire idea of relativity is that there is no such thing as a "special" or unique frame (it is like buying a "unique generic" drug!). The "special" refers to the study of relativity between inertial reference frames (ironically using Newton's law of intertia to select frames, but then essentially discarding the Newtonian framework of time and space). The formalism and predictions of special relativity can also be extended to observers in accelerated frames as long as you treat some "reference" inertial frame as a "true" frame and make predictions about the modified observations that the accelerated observers make.

   The MM experiment was designed to measure the speed of the MM apparatus (and presumably the earth that the MM apparatus was attached to) through the ether. The ether was presumed to be the medium through which light waves propagated. Hence, if you were in a frame that was at rest relative to the ether, light would travel through this "ether vacuum" at speed c as predicted by Maxwell's equations. This frame *would be* unique, special, "set aside", "perfect", etc. because this would be the frame where Maxwell's equations were true. All other frames would find a different c (like the boat in the river problem) and so they would have different, more complex Maxwell equations, that would need to include speed relative to the ether. However in the context of special relativity all laws of physics should be the same for all intertial observers, including Maxwell's equations. The equations say c and you are in an interial frame (at least approximately) you get c. (It is approximate. The surface of the earth accelerates as the earth rotates and revolves and it is also in a gravitational potential.)

3. What is the principle of equivalence in the context of general relativity? Give an example of real world system where the effects of general relativity need to be included. (2)

   Answer: The principle of equivalence states that it is impossible by any physical measurement to distinguish between a uniform gravitational field as given by g and a constant acceleration $a = -g$. In this way if you can use special relativity to make predictions about accelerated frames (see above) then you can also make predictions about non-accelerated frames in a gravitational field. So if you have an accelerated frame a clock at the "top" runs faster than the clock at the "bottom" so a clock lower in a gravitational potential well runs slower than a clock that is higher in the well.

   You need to account for this effect with the global positioning system The system depends on very accurate timing of clocks that are sent from an orbiting system of satellites to a receiver on the ground. By measuring the time interval to within a nanosecond you can find your location within 30 cm. The problem is measuring the time interval requires a clock on the satellite and a clock in the receiver; they run at different rates (and both run slow relative to a clock at infinity). In fact for every second that elapses the satellite clock gains about 1 nanosecond. Since the signal is sent a nanosecond "early" the measured interval is too short and your receiver interprets that as you approaching the satellites. In less than 5 minutes the entire result would be nonsense i.e. there isn't any point in space that could be that close to all of the satellites simultaneously).

4. A clock is at rest in reference frame S'. Consider that the frame S' is moving at $v = 0.7c$ relative to the S frame and the clock was at the origin at $t = t' = 0$. All of the usual conventions are in effect.

   (a) Draw a space-time diagram that includes the light cone, roughly correct scaling for the x' and ct' axes, and the worldline of the clock (Give yourself some space since you need to put other things on later in the question, maybe 3 units for the positive vertical and horizontal axes.) (2).

   See Figure 1 and its caption. In order to scale the primed axes remember that $x' = 1$ is always vertically above $x = \gamma$ and an entire set of $x' = 1$ points for different γ and β would trace out a hyperbola that has the light cone as its asymptote. The ct' axis and scale is symmetric with the x'-axis.

   (b) Imagine that 3.33 ns have elapsed on the clock at rest in S' and this clock emits a light flash. Is the 3.33 ns a proper time? Why or why not? (2)

   Answer: This is a proper time interval because the time interval between the two events is measured with a *single* clock (or the two events have $\Delta x' = 0$. Most of you said "it is a proper time because the clock is at rest in this frame". That is a dicey statement from the point of view of relativity because everyone disagrees on whether or not a clock is moving. Also if you measure a time interval with *two* clocks that aren't moving you won't get the proper time. The key is that $\Delta x' = 0$ so in some frame the two events occur at the same location and in the *S'* frame $\Delta t' = \tau = \Delta s'/c$.

   (c) According to observers in the S frame how much time has passed when $t' = 10$ ns (supposed to be 3.33 ns). Where is the clock at this time? Show these results on the space-time diagram. (3)

   Answer: This is where I failed to properly edit the midterm. I originally had $t' = 10$ ns throughout the question but I thought I could make the math easier by changing it to 3.33 ns (since $ct' = 1$ m).

   Answering the question as stated on the midterm: if $t' = 10$ ns and this is a proper time then according to formula 9 $\Delta t = \gamma\tau = 14$ ns. The clock is moving at $0.7c$ so its location (using standard kinematics) is $x = \beta c \Delta t = (0.7)(0.3)(14) = 2.94$ m. You could also answer the question using inverse Lorentz transforms by substituting in the values for $x' = 0$ and $t' = 10$ ns. (Remember you get the inverse Lorentz transforms from formula 3 but switching primed and unprimed variables and changing $-v$ to v.)

   You need to be a bit cautious putting this on the space-time diagram because the units are in metres. $ct' = (0.3)(10) = 3$ m. However once that is done you can draw standard Cartesian horizontal and vertical drop-lines to the locations $ct = 4.2$ m and $x = 2.94$ m (note this point does not fit on Figure 1; I chose a scale appropriate for the intended question).

   Using $t' = 3.33$ ns, $ct' = 1\ m$. $ct = 1.4$ m, $t = 4.66$ ns, $x = 0.98\ m$ (It is just a coincidence arising from $\beta\gamma \approx 1$ that $ct' \approx x$. This point is on the space-time diagram and is the

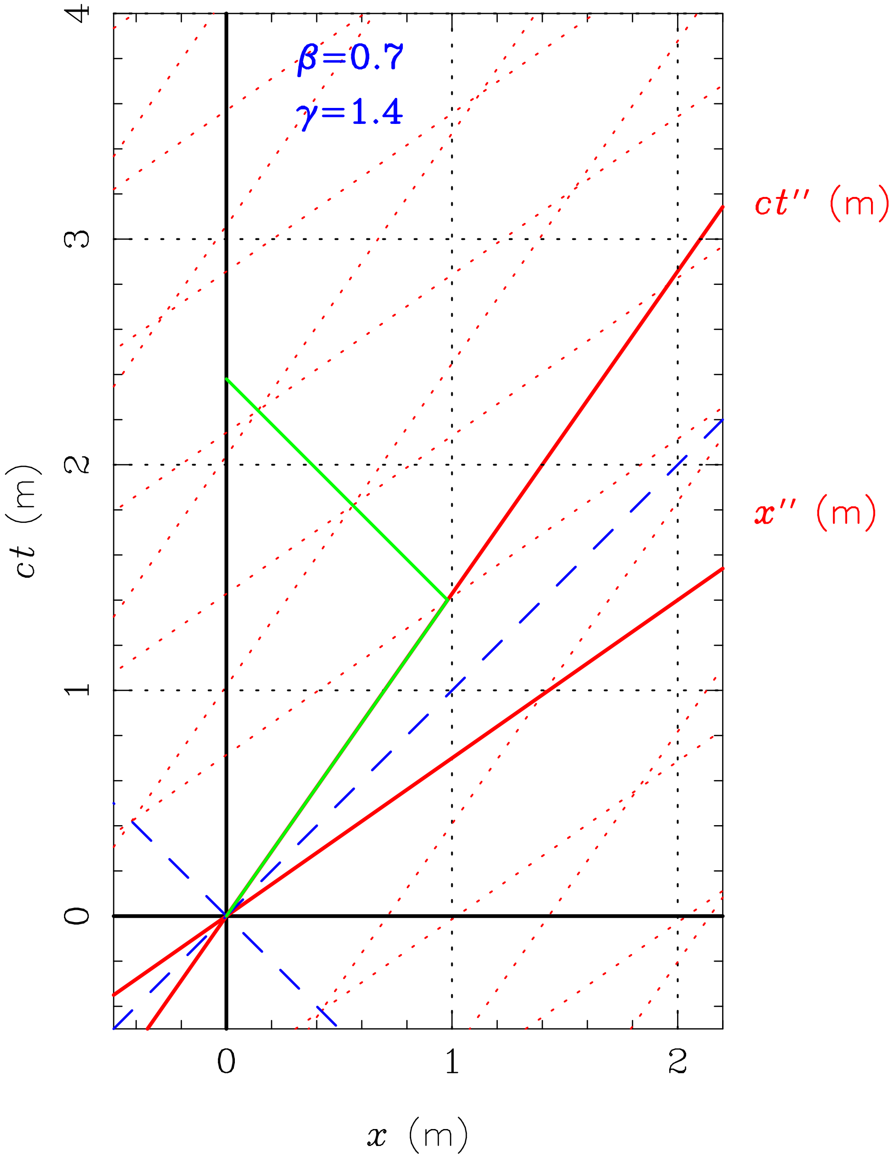

Figure 1: Spacetime diagram for Question 4. The "primed" axes are in red along with scaling lines (Sorry, had to use a double prime rather than a single prime in the diagram). The dashed red lines are the lines of constant ct' and x' set out in a 1 m by 1 m grid. The dashed blue lines are the light cone (i.e. world lines of a light flash emitted from the origin at $t = t' = 0$. The green line that is along the ct' axis is the world line of the clock and then the other green line is the world line of the light flash from the clock going back to the origin. The location $x' = 1$ m is at $x = 1.4$ m.

vertex of the two green world-lines and coincides with the intersection of the dashed red $ct' = 1$ m line and the $x' = 0$ line a.k.a. the ct'-axis.

(d) Using the results in the previous questions calculate the space-time interval $(\Delta s)^2$ in both frames using the coincidence of the origins and the emission of the signal as the two events. Is this a space-like or time-like interval? (2)

Answer: Even with my mistake if you read part (b) it mentions that the emission of the signal occurs at $t' = 3.33$ ns but when I say "use the results in the previous questions" it is really confusing. Let me answer the question the way I meant, with event 2 occuring at $t' = 3.33$ ns and $x' = 0$. Then using formula 12

$$(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = \{(0.3)(3.33)\}^2 - 0^2 = +1\,\text{m}^2 \tag{4}$$

and

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (1.4)^2 - (0.98)^2 = +0.9996\,\text{m}^2 \tag{5}$$

So the space-time interval is invariant as it must be according to formula 13.

Even if you put in $t' = 10$ ns if you keep all of correct transformed quantitities you will still get this result with $(\Delta s)^2 = 9$ m^2.

And in both cases the interval is time-like since $(\Delta s)^2$ is positive. This must be the case since in frame *S'* the two events occur at *the same place*. No observer can claim that events separated by a space-like interval ever occur at the same place.

(e) Draw the worldline of the light flash as it heads toward the origin of S. At what time t does it arrive? (Hint: between 5 and 10 ns.) Show that this time [equal to a period $T = (\nu_{\text{redshift}})^{-1}$] agrees with the prediction of the relativistic Doppler effect with $\nu_o = 300$ MHz (either symbolical or numerical proof). (3)

Answer: This is the 2nd green line in the figure. It represents a light flash going to the left so it has a slope of -1. You know the distance back to the origin in the S frame is 0.98 m and the light pulse also travels at velocity $-c$ in this frame so the additional time elapsed is $0.98/0.3 = 3.27$ ns and the total time is $4.66 + 3.27 = 7.93$ ns. You get the same result by just doing a little geometry on the space-time diagram to find the intersection of the green world-line of the light flash intersects the ct-axis at 2.38 m and then you convert that to time.

If the clock at rest in *S'* is sending flashes at a (proper) frequency of 300 MHz then the period between flashes is $T' = 3.33$ ns. So the period measured in S will be the total Δt we just calculated 7.93 ns. The redshifted frequency will be the inverse of that period $(7.93\,\text{ns})^{-1} = 0.126$ GHz= 126 MHz. Checking this with the Doppler formula (number 5 on the formula sheet)

$$\nu_{\text{redshift}} = \nu_0\sqrt{\frac{1-\beta}{1+\beta}} = 300\sqrt{\frac{0.3}{1.7}} = 126\,\text{MHz}. \tag{6}$$

5. Make a sketch of the blackbody spectrum for 3 different temperatures. What law describes the area under the spectrum as a function of temperature. At what wavelengths is the "density of modes" highest? Convert $\lambda = 450$ nm into wavevector. (5)

Answer: See Figure 2. The area under the spectrum (the total radiated intensity) is given by the Stefan-Boltzmann law (formula 23).

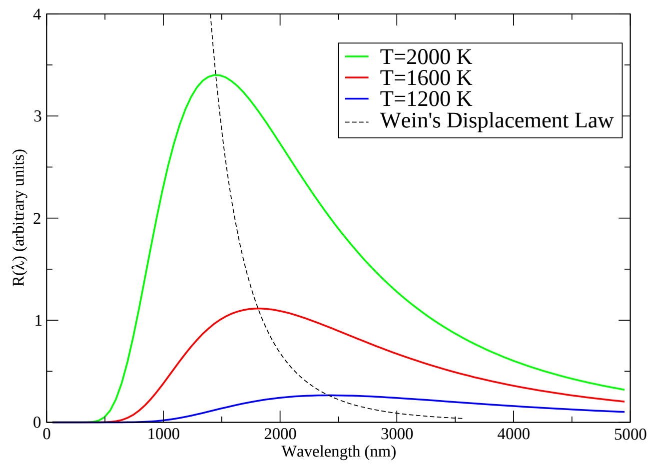

Figure 2: Blackbody spectra for Question 5. In this computer generated plot the temperature and wavelength scales are correct (but you weren't expected to include those). What I wanted to see in your answer was an increase in intensity with temperature and a trend for the peak to move to lower wavelengths at higher temperatures (as I have shown by the Wein's displacement law curve.).

I didn't give the mode density in term of wavelength but it is in terms of wavevector in formula 27 (you could use 28 to convert everything to λ) and formula 30, the Rayleigh-Jeans law, is the mode density times the equipartition result. So the mode density is highest at short wavelength.

The conversion is simply

$$k\frac{2\pi}{\lambda} = \frac{2\pi}{4.5 \times 10^{-7}\ \mathrm{m}} = 1.4 \times 10^{7}\ \mathrm{m}. \tag{7}$$

6. In the context of the discovery of the electron as a fundamental constituent of matter what kind of experiments did Faraday do? (2)

   Answer: Faraday performed *electrolysis* experiments. In these experiments electric currents were passed through different solutions and you could produce pure elements at the anode and cathode. These experiments showed

   (a) A charged, fundamental constituent of matter could pass along a wire. This component only comprised a small fraction of the mass.

   (b) That the structure of matter and chemical bonds were mitigated by electromagnetic forces.

   (c) Could also see that Faraday's constant 96500 Coulombs per mole was equal to eN_A where e was the charge of an electron.