**Solutions for Midterm Quiz: Modern Physics 301**
**November 1, 2001**

1. Value: 12 Points. An electron is traveling at a speed of $0.99c$ in the reference frame of the laboratory. We shall call the lab reference frame $S$ and the electron reference frame $S$'.

   (a) The electron travels 1 m in $S$. How much time has passed in the laboratory? How much time has passed for the electron i.e for a clock in $S$' that is fixed at the position of the electron? (2)

   **Answer:** $t = L/v = 1/[(0.99)(3 \times 10^8)] = 3.37$ ns. The time that is measured at the electron is *the proper time* $\tau$. $t = \gamma\tau$ and

$$\tau = \frac{t}{\gamma} = 3.37/\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{3.37}{7.09} = 0.475 \text{ ns} \tag{1}$$

   (b) According to an observer in $S$', how fast is the laboratory moving? Why does this not conflict with the different times that you determined in part (a)? (2)

   **Answer:** Velocity is still $0.99\,c$, just in the opposite direction. The reason it doesn't conflict is that the electron sees 1 m in the lab *Lorentz contracted* to 14 cm so velocities agree. It is also true that the relativity postulate would be violated if the answers were different.

   (c) What is the total relativistic energy of the electron in MeV? The rest mass of the electron is 0.511 MeV/$c^2$. What is the relativistic momentum of the electron? (2)

   **Answer:** $E = \gamma mc^2 = (7.07)(0.511 \text{ MeV}) = 3.62$ MeV. You can use $p = \gamma mu$ if you wish $p = (7.07)(0.511 \text{ MeV}/c^2)(0.99\,c) = 3.59$ MeV/$c$. Or use the formula I humbly referred to as **the most important formula is this section**. Just use $c = 1$ units.

$$p^2 = E^2 - m^2 \tag{2}$$

$$p = \sqrt{3.62^2 - 0.511^2} = 3.59 \text{ MeV}/c \tag{3}$$

   (d) In $S$ an anti-electron or *positron* approaches the initial electron with the same speed but in the opposite direction. According to an observer in $S$' what is the speed of the approaching positron? (2)

   **Answer:** You need to use the velocity addition formula.

$$v' = \frac{u+v}{1+uv/c^2} = \frac{0.99c + 0.99c}{1+0.99^2} = 0.999949c \tag{4}$$

   I think in solid state chemistry language when you talk about purity this would be referred to as "four and half nines".

   **So far everything in this problem is very similar to the Lenny "the Laser" assignment question.**

(e) The electron and positron collide and produce two high energy photons that have equal energy in $S$. You know that photons have no rest mass and that electrons and positrons have the same rest mass. Give the energy and magnitude of the momentum of each photon using the units we established earlier. (2)

**Answer:** In S both electron and positron have the same energy since they have equal rest masses and equal velocities. You know that the total energy is then $2 \times 3.62$ MeV. Since two photons are produced with equal energy the energy of each is just 3.62 MeV. To find the momentum just use **the most important formula in this section** with $m = 0$, $p^2 = E^2$ so $|p| = E = 3.62$ MeV/$c$. *If a particle has no rest mass (or the rest mass is small) then* $E = p$. Photons certainly have momentum; that is why they can scatter electrons, as seen in the Compton effect, which we discussed in class. Don't be concerned about the $c$ terms. You can always put them in at the end when you are using these types of units.

(f) Why were the energies of the photons equal in the previous question? Why can't there be just one photon produced in this electron-positron collision? (2)

**Answer:** Momentum conservation. We just determined that $E = p$ for photons. Suppose the photon energies, measured in $S$, were not equal. Then there must be some momentum left over. But before the collision $\Sigma\vec{p} = 0$ so there can be no momentum left over and the photons must have equal and opposite momenta in $S$. The frequencies and directions of the photons are appropriately shifted in $S$'. If there was just one photon produced, then it would have to have $p = 0$. But since $E = p$ this would imply $E = 0$, which cannot be true by energy conservation. Proper application of conservation laws is essential for determining the products in high-energy collisions.

2. Value: 8 points. Consider a blackbody at 6000 K. You may find $hc = 1240$ eV-nm and $k = 8.62 \times 10^{-5}$ eV/K useful.

(a) What is the energy of a photon with $\lambda = 590$ nm? What is the ratio of this energy to $kT$? (3)

**Answer:** $E = h\nu$ by Einstein's explanation of the photoelectric effect. In terms of wavelength $E = hc/\lambda = 1240/490 = 2.10$ eV. $kT = (8.62 \times 10^{-5})(6000\text{ K}) = 0.517$ eV. The ratio of the two is 4.06.

(b) Using Planck's formula for the energy density of a blackbody $u(\lambda)$ and your previous answer to calculate the *approximate* energy density between 580 nm and 600 nm. Give your answer in mJ/m$^3$. (5)

**Answer:** Planck's formula is a bit tough but I hoped your experience with Maxwell's velocity distribution would help. First we can recognize that $E/(kT)$ is fairly large so

$$\frac{1}{exp(\frac{hc}{\lambda kT}) - 1} \approx e^{-\frac{hc}{\lambda kT}} = e^{-4.06} = 0.017. \tag{5}$$

For those of you in thermal physics this is the *fractional occupancy* of this boson mode. Now there is a prefactor in $u(\lambda)$ of $8pihc/\lambda^5$ and we need to multiply by

20 nm (*all of you recognized this; good job!* ). Well, you know $hc/\lambda = 2.1$ eV $20/590 = 0.0339$, so that leaves:

$$u(\lambda)d\lambda = 8\pi\frac{(2.1\text{ eV})(0.0339)(0.017)}{\lambda^3} \tag{6}$$

The conversion factor from eV to mJ is $1000e$ where $e$ is electronic charge. Express $\lambda$ in metres.

$$u(\lambda)d\lambda = (25.13)\frac{(2.1)(0.0339)(0.017)}{(5.9\times10^{-7})^3}(1.6\times10^{-16}) = 24\,\text{mJ/m}^3 \tag{7}$$

I hoped that by prompting you for an answer in these units that it would act as a bit of guide to whether or not your answer was in the right ball park.

3. Value: 10 points. The probability distribution of the position of particle $f(x)$ is shaped like an isosceles triangle (think Bell curve but more "triangle"-like. Its highest point is at $x = 0$ and its lower corners, where the distribution approaches zero, are at $x = \pm W$.

   (a) What is $f(x = 0)$ such that $f(x)$ is normalized to 1? (2)
   **Answer:** Easy. The area of a triangle is $\frac{1}{2}$ base times height and it must be equal to 1. Answer is $1/W$.

   (b) What is the probability that $x > W/2$ (i.e. that the particle is found with $x > W/2$)? (3)
   **Answer:** Only slightly more difficult. It is just a different triangle. The base is $W/2$ and the height is one half of full height $1/(2W)$. Throw in the extra factor of one half for the area of a triangle and the answer is 1/8.

   (c) Calculate $(< x^2 >)^{1/2}$. Hint: the integral is symmetric about $x = 0$. (5)
   **Answer:** This one is tougher since you really need to setup an integral which requires a functional form for $f(x)$. If you had a functional form you could have answer the other two parts of the question the same way.
   We want the integral

$$< x^2 >= \int_{-\infty}^{\infty} x^2 f(x)\,dx = 2\int_0^W x^2 f(x)\,dx \tag{8}$$

where I have used the symmetry and range of the integrand to simplify the form. Over this restricted range $f(x)$ is simply a straight line. The y-intercept is $1/W$ and the slope is rise/run$=\frac{-1/W}{W} = -1/W^2$. The integral becomes

$$< x^2 >= 2\int_0^W x^2(-\frac{x}{W^2}+\frac{1}{W})\,dx \tag{9}$$

$$= \frac{2}{W}\left(-\frac{x^4}{4W}+\frac{x^3}{3}\right)|_0^W \tag{10}$$

$$= \frac{2}{W}\left(-\frac{W^4}{4W}+\frac{W^3}{3}\right) \tag{11}$$

$$= 2W^2\left(-\frac{1}{4}+\frac{1}{3}\right) = \frac{W^2}{6}. \tag{12}$$

And the answer is $W/\sqrt{6}$ when you take the square root. Pay attention to where the $W$ terms are; you should not have situations where the dimensions are different in the different terms. Also remember that the rms value is something like a half-width. You would say that the half-width of the triangle is $W/2$. Compare that to $W/2.45$ which is the right answer.

I will make some comments in class and I have spoken to several of you but here is some advice.

1. If you are really low on time to study, redo assignment questions. Not just reading the answers but typing numbers into your calculator and getting the right answers out. Try to avoid the method of reading the book but never touching a pencil, paper, or calculator unless you are really fuzzy on the basic concepts.

2. The first parts of most questions were quite easy. Make sure you have time to answer these questions properly. Also if you think the answer is easy you should know what you will get. Think about the numerical value that you write down. The old saying of a "bird in the hand is worth two in the bush" applies here.

3. When you approach a question identify the concept involved and write down the equation you will need. I gave a lot of points for just writing down the right formula and I will continue to do so in the future.