Assignment #5: PHYS 322: Electromagnetic Theory I
Due: December 2, 2009

Each question is worth 15 points. Do any two of the three but remember that you will see all of these topics on the final exam. These are questions from the Fall 2005 exam so it was expected that each question would take less than 30 minutes!

1. Consider an infinitely long square pipe whose central axis is coincident with the z-axis with sides at $x = \pm a/2$ and $y = \pm a/2$. The conducting sides at $x = \pm a/2$ are grounded and the upper side at $y = a/2$ has a potential of $V_0/2$ and the lower side at $y = -a/2$ has a potential of $-V_0/2$.

   (a) Based on the symmetry of the problem what do you conclude about the dependence of V on x, y, and z. (even, odd, constant) inside of the pipe.

   (b) Show that
   $$V(x,y,z) = \sum_{n=0}^{\infty} C_n \cos\left[(2n+1)\frac{\pi x}{a}\right] \sinh\left[(2n+1)\frac{\pi y}{a}\right]. \tag{1}$$
   satisfies Laplace's equation and the symmetry considerations in part (a).

   (c) Why have I choosen this particular form for "k"? (k is the coefficient of x or y in the trigonometric or hyperbolic function). Demonstrate mathematically what this means.

   The values of C_n are determined using Fourier series as we discussed in class. The first coefficient is
   $$C_0 = \frac{2V_0}{\pi} \frac{1}{\sinh(\pi/2)} \tag{2}$$

   (d) Determine the approximate value of $\vec{E}$ at the upper plate by just keeping the C_0 term for the solution of $V(\mathbf{r})$. What is the charge density σ on the top plate implied by this $\mathbf{E}$. (Rembember the normal vector points out from the conductor so it is $-\hat{y}$.) What would you expect for the value of $\mathbf{E}_{\text{tan}}$ at the plate to be for the full solution?

   (e) Where is the approximate solution most accurate? Least accurate? (the answer is based on a class discussion and some plots that I showed you)

2. There are two point charges of q sitting on the z-axis at $z = \pm d$. Consider using a multipole expansion to approximate $V(r,\theta,\phi)$.

   (a) Give the exact solution for $V(r,\theta,\phi)$. (Hint: if you are using Law of Cosines $\cos(\pi - \theta) = -\cos\theta$)

   (b) Calculate Q_T, $\mathbf{p}$, and $\mathcal{Q}_2$. State your answers in terms of Cartesian unit vectors or as matricies. Have another look at $\mathcal{Q}_2$ and make sure all of your factors are there!

   (c) What is $\mathbf{r}$ in terms of r, θ, ϕ, and Cartesian unit vectors?

   (d) Give the approximate solution for V up to order $1/r^3$.

3. Consider a sphere of radius a centred on the origin. The potential inside and outside the sphere is described as follows
   $$V_{\text{inside}}(r,\theta,\phi) = Ar\cos\theta \tag{3}$$
   $$V_{\text{outside}}(r,\theta,\phi) = Aa^3\frac{\cos\theta}{r^2} \tag{4}$$

where A is a constant.

(a) You know that both of these forms are separable solutions to Laplace's equation. What does that say about ρ when you are not at the surface?

(b) What fields do you get from these potentials? (just in words is fine) Sketch $\mathbf{E}$ in both regions.

(c) Is the sphere a conductor? Why or why not?

(d) Suppose that the sphere has a uniform polarisation $\mathbf{P} = 3\epsilon_0 A\hat{z}$. Calculate the bound surface charge and show that the boundary conditions for $\mathbf{E}$ are satisfied at $r = a$. (Hint: remember that $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$.) (9)

(e) There is no free charge in the problem.

   i. Does this mean that the dispacement field $\mathbf{D} = 0$?

   ii. Calculate $\mathbf{D}$ in both regions using $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$.

   iii. Where is $\nabla\times\mathbf{D} \neq 0$ (or perhaps, where are the tangential components discontinous)?

   iv. Where is $\nabla \times \mathbf{E} \neq 0$?