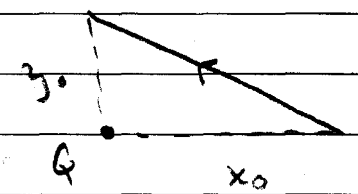
Example of a line integral



$$\Delta V = -\int_{'x_0'}^{'y_0'} \vec{E} \cdot d\vec{l}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} = \frac{Q}{4\pi\epsilon_0} \left(\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r^3} \right)$$

x-range x_0 to 0 $\quad$ s parameter

$$x(s) = x_0 - s \qquad s \in [0, x_0]$$

now if we have $y(x) = mx + b$ just sub in $x(s)$

$$y(s) = -\frac{y_0}{x_0}(x_0 - s) + y_0$$

$$= \frac{y_0}{x_0} s$$

might be times that you find functions directly $x = R\cos s$
$y = R\sin s$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \right) ds$$

$$= \left(-\hat{x} + \frac{y_0}{x_0}\hat{y} \right) ds \qquad (ds \to d\hat{l} \quad \nwarrow)$$

$$-\int_{\text{start}, P}^{\text{finish}} \vec{E}(x(s), y(s), z(s)) \cdot d\vec{l}(x(s), y(s), z(s), ds)$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{s=0}^{s=x_0} ds \left(\frac{-x(s)}{r^3} + \frac{y_0}{x_0}\frac{y(s)}{r^3} \right)$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{s=0}^{s=x_0} ds \left(\frac{s-x_0}{r^3} + \left(\frac{y_0}{x_0}\right)^2 \frac{s}{r^3} \right)$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{s=0}^{s=x_0} ds \, \frac{1}{r^3}\left[\left(1+\left(\frac{y_0}{x_0}\right)^2\right) s - x_0\right]$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{s=0}^{s=x_0} ds \, \frac{1}{\left((x_0-s)^2 + \left(\frac{y_0}{x_0}\right)^2 s^2\right)^{3/2}}\left[\left(1+\left(\frac{y_0}{x_0}\right)^2\right) s - x_0\right]$$

$$= -\frac{Q}{4\pi\epsilon_0}\int_{s=0}^{s=x_0} ds \, \frac{1}{\left(x_0^2 - 2x_0 s + \left(1+\left(\frac{y_0}{x_0}\right)^2\right)s^2\right)^{3/2}} \times$$

$$\left[\left(1+\left(\frac{y_0}{x_0}\right)^2\right) s - x_0\right]$$

need to complete the square in the denominator.

$$x_0^2 - 2x_0 s + \left(1+\left(\frac{y_0}{x_0}\right)^2\right)s^2 = \left(\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}\, s - \frac{x_0}{\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}}\right)^2$$

$$+ x_0^2 - \frac{x_0^2}{1+\left(\frac{y_0}{x_0}\right)^2}$$

$\left(\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}\, s - \frac{x_0}{\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}}\right)\sqrt{1+\left(\frac{y_0}{x_0}\right)^2}$ is the numerator leading to the "miracle substitution"

let $u = \sqrt{1 + \left(\frac{y_0}{x_0}\right)^2}\, s - \frac{x_0}{\sqrt{1 + \left(\frac{y_0}{x_0}\right)^2}}$

$du = \sqrt{1 + \left(\frac{y_0}{x_0}\right)^2}\, ds$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{s=0}^{s=x_0} du \frac{u}{(u^2 + a^2)^{3/2}} \qquad a = x_0\left(\frac{y_0}{\sqrt{x_0^2 + y_0^2}}\right)$$

$$= -\frac{Q}{4\pi\epsilon_0} \frac{-1}{\sqrt{u^2 + a^2}} \Bigg[_{s=0}^{s=x_0}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r(s)} \Bigg[_{s=0}^{s=x_0}$$ (recognizing $r = \sqrt{u^2 + a^2}$ is a big help)

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{\left((x_0 - x_0)^2 + \frac{y_0^2}{x_0} x_0^2\right)^{1/2}} - \frac{1}{\left(x_0^2 + \frac{y_0^2}{x_0} 0^2\right)^{1/2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{y_0} - \frac{1}{x_0}\right)$$

wow, very single result.
This indicates something special about $\vec{E}$ and line integrals.

Wonder if this depends on the path?