Physics 322: Notes on the Helmholtz Theorem
Sept. 2009

In order to change variables on the "del" operator ∇ when it acts on the $1/\imath$ scalar field just work out both ∇ and ∇' expressions and compare them

$$\nabla\left(\frac{1}{4\pi\,\imath}\right) = \nabla\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = -\frac{1}{4\pi}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\vec{\imath}}{4\pi\,\imath} \tag{1}$$

and

$$\nabla'\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = \nabla'\left(\frac{1}{4\pi|\mathbf{r}'-\mathbf{r}|}\right) = -\frac{1}{4\pi}\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3} = -\left\{-\frac{1}{4\pi}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right\} = -\nabla\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) \tag{2}$$

since $|\mathbf{r}-\mathbf{r}'| = |\mathbf{r}'-\mathbf{r}|$.