Physics 322: Getting the correct sign in line integrals
October 25, 2006

The context of this problem is that I was attempting to determine the capacitance per unit length between two coaxial conductors. The inner one has radius a and the outer one has radius b. We will make the assumption that the conductors are infinitely long and have axes parallel to the z-axis. For this problem I will use (s, ϕ, z) as cylindrical polar coordinates (consistent with Griffiths, *Introduction to Electrodynamics* 3rd ed.). This problem appears as 2.39 in Griffiths. From a practical standpoint I thought this problem was interesting because every student has used a coaxial connection to an oscilloscope to measure voltage and this is a way to estimate how much charge it takes to maintain a voltage between the leads. From a teaching point of view I thought that I would work through something other than the canonical parallel plate capacitor. I will let the students do that on their own.

1. Since the students have just been introduced to solving Poisson's (or Laplace's) equation and matching boundary conditions I am approaching this from a Gauss Law point of view.

2. The symmetry permits the following simplifications to the electric field:

$$\vec{E}(\vec{r}) = \vec{E}(s, \phi, z) = \vec{E}(s) = E_s(s)\hat{s} \tag{1}$$

   The problem has azimuthal symmetry and infinite translational symmetry along z so we expect no ϕ or z dependence. We expect no ϕ component since that would necessarily "curl" around the wire [I have worked this through for the students. They know that if the ϕ component varied as $\frac{1}{s}$ then the curl would be zero in that region (except for $s = 0$), so there is a possible exception. But it would still be inconsistent with no tangential component in the electric field at the surface of the conductors.] There is no z component because this would break the reflection symmetry of the infinitely long wire. (It would pick an "up" or "down". Students also seem comfortable with saying that in a Coulomb's law integral that elements of charge at z and $-z$ will cancel each other's z-component.)

   Caveat: I did this in the context of electric field but in fact these arguments only hold if the distribution of *charge* has azimuthal and translation symmetry. This is not necessarily guaranteed with conductors since the charge flows to create an equipotential surface and there is nothing else to break the symmetry. But as an example if you had a cylindrical inner conductor and a square outer conductor you would have to admit some ϕ dependence.

3. Use a Gaussian surface of a cylinder of radius a and length ℓ. The charge per unit length on the inner conductor is λ (or per unit area $\sigma = \frac{\lambda}{2\pi a}$). The outer conductor has $-\lambda$ to guarantee that the field vanishes for $s > b$ (where you would be in the bulk of the outer conductor). The charge enclosed by the cylindrical Gaussian surface is $\lambda\ell$. The flux integral vanishes at the top and bottom of the cylinder, and is normal to and constant upon the walls of the cylinder. Left and right sides of Gauss' Law are then

$$E_s(s) 2\pi s\ell = \frac{\lambda\ell}{\epsilon_0} \tag{2}$$

   and the electric field in the $a \le s < b$ region is

$$\vec{E}(\vec{r}) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}. \tag{3}$$

Just like the result for an infinitely long wire. (I pointed out the analogy to the relationship between the electric field of a point charge and spherical shell.) This form of the electric field is completely consistent with the boundary conditions for the electric field at the surface of the conductor.

4. If I want capacitance I need to calculate the potential between the two conductors. The prescription to find the potential between points $\vec{A}$ and $\vec{B}$ is to perform a line integral

$$V(\vec{B}) - V(\vec{A}) = -\int_C \vec{E} \cdot d\vec{\ell} \tag{4}$$

along a curve C such that the positive direction runs from $\vec{A}$ to $\vec{B}$. This is the direction of $d\vec{\ell}$. This direction or orientation must be defined for the integral to have meaning. Since $\nabla \times \vec{E} = 0$ we know that this potential difference depends only on the end points and the direction, not on the details of the path. Griffiths says that for capacitors one should travel from the conductor at low potential (and with the negative charge) to the conductor with high potential (the one with positive charge). So the potential difference *must* be positive. Indeed, if one draws a path going from $s = b$ to $s = a$ the direction of $d\vec{\ell}$ is inward, the electric field is outward, the dot product of the two is always negative, the integral (which is just supposed to be a sum) is negative, and the value of the total (with the negative sign out front) is positive. You could imagine moving a positive test charge from $s = b$ to $s = a$ and it would take work to do so. And the work per unit charge would give you potential difference. **Students:** This is a line of reasoning based on physics as opposed to just mathematics. It is the type of argument you should be able to make and will always lead you to the correct answer.

5. So it is the math that gets me into trouble. Since the path is oriented inward I said that in terms of our parameter s, $d\vec{\ell} = -\hat{s}ds$. And the limits of integration are a and b. Since I am going from b to a I reasoned that b should be the lower limit of integration and a the upper.

$$\begin{aligned} V(s=a) - V(s=b) &= -\int_{s=b}^{s=a} \left(\frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{s} \right) \cdot (-\hat{s}ds) && (5) \\ &= \frac{\lambda}{2\pi\epsilon_0} \int_b^a \frac{ds}{s} && (6) \\ &= \frac{\lambda}{2\pi\epsilon_0} (\ln a - \ln b) && (7) \\ &= -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right) && (8) \end{aligned}$$

which is definitely negative. Where did I go wrong? (and I am *wrong*, remember the physics argument!)

6. **The Griffiths option** According to Griffiths, where I have found similar examples (Example 1.6, Example 2.6), you are not permitted to change the sign of $d\vec{\ell}$ and according to the front cover $d\vec{\ell} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}$. This works mathematically because I eliminate one of the negative signs but I don't like it because typically everyone *assumes* that differentials are positive (they are the widths of the Reimann sum rectangles) so one would conclude here that $d\vec{\ell}$ is pointing the wrong way. But, it does give you the right sign, every time.

7. **The sign of the differentials** The assumption that the differential is positive is not necessarily a bad thing but it must be enforced i.e. the limits of integration must be increasing.

The parameter we use to describe paths always comes back to or is based on the idea of path or arc length which should *increase* as we move along the path, even if the coordinates (s in this case) are decreasing. In this sense s is clearly not path length; its derivative has the wrong sign and it doesn't have the right limits. We could define a parameter t that was path length and does increase from 0 to $b-a$. It would be related to s as follows:

$$\begin{aligned} s &= b - t && (9) \\ ds &= -dt && (10) \\ d\vec{\ell} &= \hat{s}\,ds = -\hat{s}\,dt && (11) \\ V(t=b-a) - V(t=0) &= -\int_{t=0}^{t=b-a}\left(\frac{1}{2\pi\epsilon_0}\frac{\lambda}{b-t}\hat{s}\right)\cdot(-\hat{s}dt) && (12) \\ &= \frac{\lambda}{2\pi\epsilon_0}\int_0^{b-a}\frac{dt}{b-t} && (13) \\ &= \frac{\lambda}{2\pi\epsilon_0}(-\ln(b-b+a)+\ln b) && (14) \\ &= \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) && (15) \end{aligned}$$

We can see now why the two methods work. The Griffiths method keeps the sign change in the limits of integration. "My" way has a more natural direction for $d\vec{\ell}$ but admittedly is more cumbersome (and there are lots of times you have a decreasing parameter).

8. **My explanation:** It is the $d\vec{\ell}$ that is tricky. Consider this example. Suppose you have an electric field $\vec{E} = E_0\hat{z}$ (a field you would see inside a parallel plate capacitor with plate separation D). If we move from the upper plate to the lower plate against the electric field we have increased the potential. Since $\vec{E}$ is constant we can take it out of the integral.

$$\Delta V = -\int_{z=D}^{z=0} E_0\hat{z}\cdot d\vec{\ell} = -E_0\cdot\left(\int_{z=D}^{z=0} d\vec{\ell}\right) \tag{16}$$

Now that integration operation is just adding up all the little bits of path and *must* be $-D\hat{z}$, giving you the expected $\Delta V = E_0 D$. But if you say that $d\vec{\ell} = -\hat{z}\,dz$ you get the wrong answer if you use the integration limits suggested by the problem. **Ah-ha!** Now we can see what is wrong very clearly. Although I try to keep the students from saying this (because they ignore the definition in terms coordinate transforms), a vector (especially a differential one) is magnitude and direction. I have picked the direction (by the orientation of the path), the magnitude cannot, must not, become negative so in fact

$$d\vec{\ell} = -\hat{z}|dz| \tag{17}$$

Now we look at the limits of integration which imply a decreasing z so we can either flip the integration limits and $|dz| = dz$ or keep the integration limits and $|dz| = -dz$. I think that this make the most sense for students. Use geometry to pick the natural direction for $d\vec{\ell}$ but ensure that the integration limits are in order of increasing parameter (even if you appear to be moving backwards along the path). You are doing the correct thing; you are just performing the addition of the Reimann sum from the correct side. There is nothing wrong with the Griffiths way either; you just have to remember that the sign of the differential (or order of limits of integration) contains some information about the direction of $d\vec{\ell}$ and you shouldn't try to put in extra signs to "correct" $d\vec{\ell}$ assuming the differentials will be positive automatically.

9. As far as the actual answer goes, we need a capacitance per unit length C', so charge per unit length naturally enters the definition

$$C' = \frac{C}{L} = \frac{Q}{VL} = \frac{\lambda}{V} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \tag{18}$$

Putting in $a = 0.5$ mm and $b = 2.5$ mm we have a capacitance per metre of 0.185 pF. I mentioned to the students that this is a case where we probably want a very low capacitance so the voltage builds up with very little charge flow. I can easily see extending this problem to ask about the forces between the conductors, energy content, etc., It would also handle a dielectric fairly well and you could use Gauss' Law for materials.