Physics 322: Example of multipole expansion

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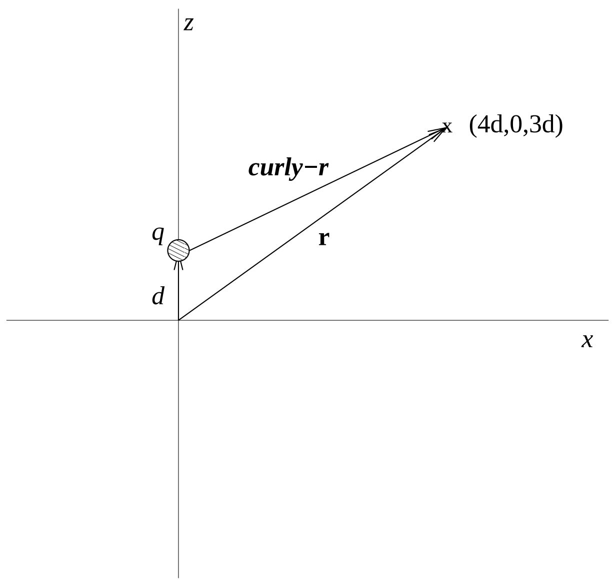

All distances in this problem are scaled by d.
The source charge q is offset by distance d along the z-axis.

1. Let's start by calculating the exact potential at the field point $\mathbf{r} = 4d\hat{x} + 3d\hat{z}$. This is easy to do since there is only one source charge.

$$V(4d, 0, 3d) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} = \frac{q}{4\pi\epsilon_0 d}\frac{1}{\sqrt{20}} \approx 0.223607\ V_0 \tag{1}$$

With $V_0 = \frac{q}{4\pi\epsilon_0 d}$ as a convenient unit of voltage.

2. As I said in class the beauty of the multipole expansion is that we attribute Q_T, $\mathbf{p}$, and $\mathcal{Q}_2$ to properties of the charge distribution in the same way we would give our own height, weight, and shoe size; they don't depend on the field point. Also when they are complete they contain information about the source coordinates *but* they don't contain the source coordinates as direct variables. Now with just one charge in the distribution I would not recommend using a multipole expansion to approximate $V(\mathbf{r})$ (i.e. you can easily write the closed, exact expression) so the steps I am showing here are merely meant to illustrate how you would do this for general charge distributions.

   (a) The total charge Q is simply q.

(b) The dipole moment

$$\mathbf{p} = \sum_k q_k \mathbf{r}_k = qd\hat{z} \tag{2}$$

The sum k is over all of the charges in the array. There are analogous expressions for line charges, surface charges, and general $\rho(\mathbf{r}')$ that involve integrals rather than sums.

(c) The quadrupole moment tensor

$$\mathcal{Q}_2 = \sum_k \frac{q_k}{2}\left(3\mathbf{r}_k\mathbf{r}_k - r_k^2\hat{I}\right) = \frac{qd^2}{2}\left(3\hat{z}\hat{z} - \hat{I}\right) \tag{3}$$

We can also represent the quadrupole moment tensor as a matrix where we multiply column vector $\hat{z}$ by row vector $\hat{z}$ rather than the opposite order (which would give a dot product).

$$\mathcal{Q}_2 = \frac{qd^2}{2}\left[3\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\right] = \frac{qd^2}{2}\begin{pmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 2\end{pmatrix} \tag{4}$$

So which one is correct expression 3 or 4? Both are equally correct since all we are trying to do is "represent" a geometrical, mathematical object. (i.e. five, V, $\sqrt{25}$, $\cancel{||||}$, a hand held up in the air with fingers spread apart all represent '5' the integer that is one bigger than four and one smaller than six but one of them is good for Superbowls and movie sequels while another is great for keeping count).

3. Now let's calculate the separate monopole, dipole, and quadrupole contributions to the potential.

The multipole expansion is

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\left(\frac{Q_T}{r} + \frac{\hat{r}\cdot\mathbf{p}}{r^2} + \frac{\hat{r}\cdot\mathcal{Q}_2\cdot\hat{r}}{r^3}\right) \tag{5}$$

Notice that this expression contains $\mathbf{r}$ only. There is neither $\cancel{\hat{z}}$ nor any source coordinates.

(a) First we need $\hat{r}$

$$\hat{r} = \frac{\mathbf{r}}{r} = \frac{4d\hat{x} + 3d\hat{z}}{\sqrt{(4d)^2 + (3d)^2}} = 0.8\hat{x} + 0.6\hat{z} \tag{6}$$

(b) The monopole contribution is

$$V_{\text{mono}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q_T}{r} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(4d)^2 + (3d)^2}} = 0.200000\ V_0 \tag{7}$$

Wow, we are already pretty close. The quality of the monopole approximation should go as roughly the size of the charge distribution divided by the distance to the field point. In this case this is $d/5d$ or 20%. This looks reasonable.

(c) The dipole contribution is

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\mathbf{p}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(0.8\hat{x} + 0.6\hat{z})\cdot qd\hat{z}}{25d^2} == \frac{q\cancel{d}}{4\pi\epsilon_0}\frac{0.6\hat{z}}{25d^{\cancel{2}}} == 0.024000\ V_0 \tag{8}$$

A bit improvement. The percentage difference between the multipole approx with two terms and the actual answer is 0.18 %. This is actually way better than we would expect! (and should have been my first clue to the problems I would enounter later). We would expect it to be as least as good as 4% (square of the size/distance ratio).

(d) The quadrupole contribution is

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\mathcal{Q}_2\cdot\hat{r}}{r^3} \tag{9}$$

Let's use the matrix multiplication technique first. The first $\hat{r}$ is represented as a row matrix and the second as a column matrix. The idea is that V_{quad} must be a scalar

$$\begin{aligned}
V_{\text{quad}} &= \frac{q\cancel{d^2}}{4\pi\epsilon_0}\frac{1}{2\times 125}\cancel{d^3}\begin{pmatrix} 0.8 & 0 & 0.6 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0.8 \\ 0 \\ 0.6 \end{pmatrix} \\
&= \frac{1}{250}\begin{pmatrix} 0.8 & 0 & 0.6 \end{pmatrix}\begin{pmatrix} -0.8 \\ 0 \\ 1.2 \end{pmatrix} V_0 \\
&= \frac{-0.64 + 0.72}{250} V_0 = 0.00032\, V_0
\end{aligned} \tag{10}$$

As long as we realize that $\hat{r}\cdot\hat{I}\cdot\hat{r} = 1$ and remember to do the dot products from the front and back we can also use expression 3

$$\begin{aligned}
V_{\text{quad}}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\frac{qd^2}{2}\frac{\hat{r}\cdot(3\hat{z}\hat{z} - \hat{I})\cdot\hat{r}}{(5d)^3} \\
&= \frac{(0.8\hat{x} + 0.6\hat{z})\cdot(3\hat{z}\hat{z} - \hat{I})\cdot(0.8\hat{x} + 0.6\hat{z})}{2\times 125} V_0 \\
&= \frac{3\times 0.6\times 0.6 - 1}{250} V_0 = 0.00032\, V_0
\end{aligned} \tag{11}$$

So the 3-term multipole expansion now gives us $V \approx 0.200000 + 0.024000 + 0.00032 = 0.22432 V_0$. Hmm, but this is slightly worse than a 2-term expansion (but still pretty good 0.33%). More confusing is that if we *subtract* the quadrupole term we get $V \approx 0.2236800$ which is almost exactly the exact answer given in equation 1 (within 0.033%). Arrgh!

4. The problem is that I have choosen a location in space where the magnitude of the quadrupole correction is very small (I did know that going in) and where the octupole correction is roughly twice as big in magnitude and opposite in sign (that was a surprise)! You probably don't want to work with third-rank tensors or go back and work out the other terms in the $1/\cancel{z}$ expansion so I will use a shortcut to figure out the octupole moment that works when the charges are distributed along the z-axis.

   (a) Recognize that $\hat{r}\cdot\hat{z} = \cos\theta$ if we give $\mathbf{r}$ in (r,θ,ϕ). This allows us to replace the dot products in V_{dip} and V_{quad} to give

$$V(r,\theta,\phi)_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}(1) \tag{12}$$

$$V(r,\theta,\phi)_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qd}{r^2}(\cos\theta) \tag{13}$$

$$V(r,\theta,\phi)_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{qd^2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) \tag{14}$$

   Hey, those terms in brackets are $P_\ell(u = \cos\theta)$. This is not an accident. We have tried to rank the contributions to the potential in terms of powers of r so each term *should* be a separable solution to Laplace's equation.

(b) So with this in mind

$$V(r,\theta,\phi)_{\rm oct} = \frac{1}{4\pi\epsilon_0}\frac{qd^3}{r^4}\frac{1}{2}(5\cos^3\theta - \cos\theta) \tag{15}$$

since $P_{\ell=3}(u) = \frac{1}{2}(5u^3 - u)$. (This isn't true for all charge distributions; just this one). In this case $\cos\theta = 0.6$

$$V(r,\theta,\phi)_{\rm oct} = \frac{1}{4\pi\epsilon_0}\frac{qd^3}{(5d)^4}\frac{1}{2}(5(0.6)^3 - 0.6) = -0.000576\ V_0 \tag{16}$$

(c) Now $V \approx 0.200000 + 0.024000 + 0.00032 - 0.000576 = 0.223744$, a percentage difference of 0.06% from the exact answer. We expect something on the order of $(d/r)^4 = 0.16\%$ so we are still doing better than expected.

(d) The lesson here. There are times that because of the charge distribution or because of the field point that the "poles" can get out of order. If you look at the expression in terms of θ you can see where the "nodes" are. Even a clock that doesn't work is *exactly* right twice a day.