Supplemental Notes: Unit Vectors

Physics 322: Electromagnetic Theory I, Carl Adams, Fall 2009

What is a unit vector? The *direction* associated with a change in a variable while other held constant. Also the *length* of the unit vector should be *unity*. Length would be calculated using the square root of the dot product. What we are interested in is how the unit vectors in one coordinate system (primed $\hat{\mathbf{e}}'_i$) are related to the unit vectors in another coordinate system (unprimed $\hat{\mathbf{e}}_j$). Primarily I am interested in transforms between Cartesian, spherical polars, and cylindrical polar coordinates i.e. the underlying space is 3-D flat Euclidean and the unit vectors are orthonormal to each other.

Consider first the case of trying to determine $\hat{\theta}$ in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$. Consider the effect of a change of dr, $d\theta$, and $d\phi$ in dx, dy, and dz.

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \quad (1)$$

$$dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \quad (2)$$

$$dz = \cos\theta\, dr - r\sin\theta\cos\phi\, d\theta. \quad (3)$$

The coefficients of the differentials are simply the partial derivatives $\frac{\partial x}{\partial r}$ etc.

So if r and ϕ are held constant we are moving in the $\hat{\theta}$ direction. We can construct a unit vector then using $d\boldsymbol{\theta} = |d\boldsymbol{\theta}|\hat{\theta}$.

$$d\boldsymbol{\theta} = dx\hat{x} + dy\hat{y} + dz\hat{z} \quad (4)$$

$$= (r\cos\phi\cos\theta\hat{x} + r\sin\phi\cos\theta\hat{y} - r\sin\theta\hat{z})d\theta \quad (5)$$

$$|dth\vec{e}ta| = \sqrt{d\boldsymbol{\theta}\cdot d\boldsymbol{\theta}} \quad (6)$$

$$= \left(r^2\cos^2\phi\cos^2\theta + r^2\sin^2\phi\cos^2\theta + r^2\sin^2\theta\right)^{\frac{1}{2}} \quad (7)$$

$$= r \quad (8)$$

$$\hat{\theta} = \frac{d\boldsymbol{\theta}}{|d\boldsymbol{\theta}|} = \cos\phi\cos\theta\hat{x} + \sin\phi\cos\theta\hat{y} - \sin\theta\hat{z} \quad (9)$$

So for the general case $\hat{\mathbf{e}}_i \rightarrow \hat{\mathbf{e}}'_j$

$$\hat{\mathbf{e}}'_j = \frac{\left(\frac{\partial x_i}{\partial x'_j}\,\hat{\mathbf{e}}_i\right)}{\left(\frac{\partial x_i}{\partial x'_j}\frac{\partial x_k}{\partial x'_j}\,\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}'_k\right)} \quad (10)$$