Chemistry 331 **Assignment #3** (due Thursday 03 October, in class)

Q1. Emissivity (energy radiated per unit area per unit time) has units J m^{-2} s^{-1}. But Planck's expression for the blackbody emissivity spectrum

[1]
$$e(T,\nu) = \frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

has units J m^{-2}. Explain this apparent contradiction.

Q2. Calculate the total emissivity of the surface of the sun, assumed to be a blackbody at 5600 K. [1]

Q3. At high frequencies ($e^{h\nu/kT} >> 1$), the blackbody spectrum simplifies to

$$e(T,\nu) = \frac{2\pi h\nu^3}{c^2} e^{-h\nu/kT}$$

[3] Use this expression $e(T,\nu)$ to estimate the emissivity of the sun in the ultraviolet region of the spectrum (wavelengths < 400 nm), an important consideration for sun burns.

useful integral: $$\int x^3 e^{ax} dx = \frac{e^{ax}}{a^4}(a^3x^3 - 3a^2x^2 + 6ax - 6)$$

Q4. Wein's law gives $\lambda_{max} = \dfrac{hc}{4.965kT}$ for the wavelength of radiation at the maximum in the emissivity of a blackbody. In what region of the electromagnetic spectrum (radio, infrared,

[4] visible, ultraviolet, X ray or gamma ray) does the maximum in the emissivity occur if:

a) $T = 2.7$ K (the temperature of cosmic background radiation)
b) $T = 290$ K (room temperature)
c) $T = 5600$ K (the temperature of the surface of the sun)
d) $T = 10^8$ K (the temperature of a thermonuclear explosion)

radiation type	wavelengths
radio	> 1 mm
infrared	1 mm to 700 nm
visible	700 nm to 400 nm
ultraviolet	400 nm to 10 nm
X ray	10 nm to 0.01 nm
gamma ray	< 0.01 nm

... page 2

Q5. Show $x(t) = C_0 + C_1 t + C_2 t^2$ is a solution of the differential equation

$$\frac{d^2x(t)}{dt^2} = C_3$$

[2] where C_0, C_1, C_2 and C_3 are constants.

Q6. A baseball is thrown straight upward with initial velocity 25 m s^{-1}. Ignoring air friction (a good approximation), Newton's law $\boldsymbol{ma} = \boldsymbol{F}$ gives the equation of motion

$$m\frac{d^2x(t)}{dt^2} = -mg$$

[4] m is the mass of the baseball, $x(t)$ is the height of the baseball at time t, and g is the gravitational acceleration (9.81 m s^{-2}).

a) Use the initial conditions $x = 0$ at $t = 0$ and $dx/dt = 25$ m s^{-1} at $t = 0$ to determine the height $x(t)$ of the baseball as a function of the time. *Hint*: See Question 5.

b) Calculate the maximum height the baseball reaches.

c) How long will it take the baseball to return to $x = 0$ from where it was thrown?

Chem 331 Assignment #3

(Q1) Planck's emissivity spectrum $\frac{2\pi h\nu^3}{c^2}\left(\frac{1}{e^{h\nu/kT}-1}\right)$ no units

has the same units as $h\nu^3/c^2$

$$\frac{h\nu^3}{c^2} \sim \frac{(J\,s)(s^{-1})^3}{(m\,s^{-1})^2} \sim \frac{J}{m^2}$$

but $e(T,\nu)$ is the distribution function for the emissivity over different frequencies, not the emissivity

the emissivity over the frequency range ν_1 to ν_2 is:

$$\int_{\nu_1}^{\nu_2} \underbrace{\frac{2\pi h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}}_{\text{units}=\frac{J}{m^2}}\,\underbrace{d\nu}_{\frac{1}{s}} \sim \frac{J}{m^2}\frac{1}{s}$$

in units of energy per unit area per unit time.

(Q2) total emissivity (integrate over all frequencies) $= \int_0^\infty e(T,\nu)\,d\nu = \sigma T^4$

$\sigma = 5.670\times10^{-8}\ J\,m^{-2}\,s^{-1}\,K^{-4}$ (Stefan constant)

for the sun ($T = 5600\,K$) $\sigma T^4 = 55.76\times10^{6}\ \frac{J}{m^2\,s} = 55.76\ \frac{MW}{m^2}$

high energy photons, can be damaging to biological cells

(Q3) ultraviolet ($\lambda^* < 400$ nm) emissivity of the sun

for frequencies $> \dfrac{c}{\lambda^*} = \dfrac{2.998 \times 10^8 \text{ m s}^{-1}}{400 \times 10^{-9} \text{ m}}$

$$\nu > \nu^* = 7.50 \times 10^{14} \text{ s}^{-1}$$

ultraviolet emissivity $= \displaystyle\int_{\nu^*}^{\infty} e(T,\nu)\, d\nu$

(for convenience, define $x = \dfrac{h\nu}{kT}$)

$$= \int_{\nu^*}^{\infty} \frac{2\pi h\nu^3}{c^2} e^{-h\nu/kT}\, d\nu$$

$$= \frac{2\pi}{c^2} \frac{(kT)^4}{h^3} \int_{\nu^*}^{\infty} \frac{h^2 h\nu^3}{(kT)^3} e^{-h\nu/kT}\, d\!\left(\frac{h}{kT}\nu\right)$$

$$= \frac{2\pi k^4 T^4}{c^2 h^3} \int_{x^*}^{\infty} x^3 e^{-x}\, dx \quad \text{(changed variable)}$$

ultraviolet emissivity $= \dfrac{2\pi k^4 T^4}{c^2 h^3} \Big[\underbrace{e^{-x}(-x^3 - 3x^2 - 6x - 6)}_{\text{(from the provided integral, with } a=-1)} \Big] \Big|_{x^*}^{\infty}$

$$\left(x^* = \frac{h\nu^*}{kT} = \frac{(6.626 \times 10^{-34})(7.50 \times 10^{14})}{(1.381 \times 10^{-23})(5600)} = 6.425 \right)$$

(Q3 cont.)

$$\text{ultraviolet emissivity} = \frac{2\pi k^4 T^4}{c^2 h^3}\left[e^{-x}(-x^3-3x^2-6x-6)\right]_{x=\infty}$$

$$- \frac{2\pi k^4 T^4}{c^2 h^3}\left[e^{-x}(-x^3-3x^2-6x-6)\right]_{x=6.425}$$

$$= \frac{2\pi k^4 T^4}{c^2 h^3}\left[0 + e^{-x}\overbrace{(x^3+3x^2+6x+6x)}^{433.6}\right] \text{evaluated at } x = 6.425$$

$$= \frac{2\pi k^4 T^4}{c^2 h^3}\, 0.7027$$

$$= \left(8.596 \times 10^6 \frac{J}{m^2 s}\right) 0.7027$$

$$= 6.04 \times 10^6 \frac{J}{m^2 s}$$

$$= 6.04 \frac{MW}{m^2}$$

about 11% of the total radiation emitted from the sun is in the ultraviolet region of the spectrum

thank goodness:
most is blocked by the ozone layer in the upper atmosphere

Q4 a) $T = 2.7\ K$

$$\lambda_{max} = \frac{hc}{4.956\, kT}$$

$$= \frac{(6.626\times10^{-34}\ J\,s)(2.998\times10^{8}\ m\,s^{-1})}{4.956\,(1.381\times10^{-23}\ J\,K^{-1})}\ \frac{1}{T}$$

$$= \frac{0.002902\ m\ K}{2.7\ K} = 0.00107\ m$$

$$= 1.07\ mm$$ Radio

b) $T = 290\ K$ $\qquad \lambda_{max} = \frac{0.002902}{290}\ m$

$\lambda_{max} = 0.0100\ mm = 10{,}000\ nm$ IR

c) $T = 5600\ K$ $\qquad \lambda_{max} = \frac{0.002902}{5600}\ m$

$\lambda_{max} = 518\ nm$ Visible

~ green light

(no accident chlorophyll absorbs green light!)

d) $T = 10^{8} K$ $\qquad \lambda_{max} = 2.90\times10^{-11}\ m$

$= 0.0290\ nm$

X-Ray

Q5 Is $c_0 + c_1 t + c_2 t^2$ a solution of the differential equation

$$\frac{d^2 x(t)}{dt^2} = c_3 \qquad \left(\begin{array}{c} c_0, c_1, c_2, c_3 \\ \text{are constants} \end{array} \right)$$

$$LS = \frac{d^2 x(t)}{dt^2}$$

$$= \frac{d}{dt} \frac{d}{dt} \left(c_0 + c_1 t + c_2 t^2 \right)$$

$$= \frac{d}{dt} \left(c_1 + 2c_2 t \right)$$

$$= 2c_2 \quad (\text{a constant})$$

$$RS = c_3 \quad (\text{a constant})$$

$\Rightarrow$ $c_0 + c_1 t + c_2 t^2$ is a solution of the differential equation with:

$$2c_2 = c_3$$

(Q6) A baseball is thrown vertically upward (in the positive x direction) with initial velocity $v_x(t=0) = 25\ \mathrm{m\,s^{-1}}$.

Newton's law: $ma_x = F_x$

acceleration $a_x = \frac{dv_x}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2}$

gravitational force ("down") (−x direction) $F_x = -mg$

this gives $m\frac{d^2x(t)}{dt^2} = -mg$ = a constant

from Question 5, the general solution of this differential equation is

$$x(t) = C_0 + C_1 t + C_2 t^2$$

evaluate C_0: at $t=0$, $x(0)=0$: $x(0) = 0 = C_0 + C_1(0) + C_2(0)^2$

$\therefore \boxed{C_0 = 0}$

eval. C_1: at $t=0$, $v_x(t=0) = \left.\frac{dx}{dt}\right|_{t=0} = C_1 + 2C_2 t$

$v_x(0) = 25\ \mathrm{m\,s^{-1}} = C_1 + 2C_2(0)$

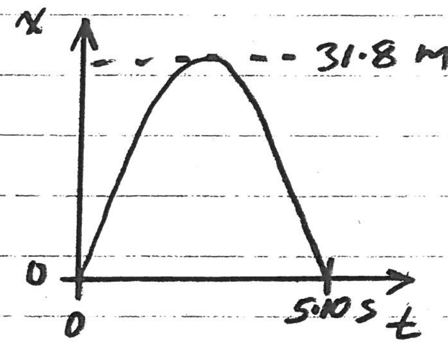
$\boxed{25\ \mathrm{m\,s^{-1}} = C_1}$

(Q6 cont.)

$$-mg = m\frac{d^2x(t)}{dt}$$

$$= m\frac{d}{dt}\frac{d}{dt}\left(C_0 + C_1 t + C_2 t^2\right)$$

$$= m\frac{d}{dt}\left(C_1 + 2C_2 t\right)$$



$$-mg = m\,2C_2$$

$$\boxed{\frac{-g}{2} = C_2}$$

$$\boxed{x(t) = (25\ m\,s^{-1})t - \frac{g}{2}t^2}$$

b) max. height when $\frac{dx(t)}{dt} = 0 = 25\frac{m}{s} - gt$

$$t = \frac{25\ ms^{-1}}{9.81\ ms^{-1}} = 2.55\ s$$

$$x_{max} = (25\ ms^{-1})(2.55\ s) - \frac{9.81\ ms^{-2}}{2}(2.55\ s)^2 = \boxed{31.8\ m}$$

c) returns to ground ($x=0$) when

$$0 = (25\ ms^{-1})t - \frac{g}{2}t^2 \qquad 25\ ms^{-1} = \frac{g}{2}t$$

$$t = \frac{(25\ ms^{-1})(2)}{9.81\ ms^{-2}} = \boxed{5.10\ s}$$