The eternal domination number for $3 \times n$ grid graphs

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Abstract

In the eternal dominating set problem, guards form a dominating set on a graph and, at each step, a vertex is attacked. To defend against the attack, each guard either remains in place or moves to a neighboring vertex in order to form a new dominating set that contains the attacked vertex. We wish to determine the minimum number of guards required to successfully defend against any possible sequence of attacks, the eternal domination number. This number is known for $3 \times n$ grid graphs when $n < 26$. This paper determines exact values of eternal domination numbers for $3 \times n$ grid graphs when $n \geq 26$.

1 Introduction

A dominating set for a graph is a positioning of guards on vertices so that every vertex is monitored from a distance of at most one. A graph's domination number is the smallest size of such a set. An eternal dominating family is a collection of dominating sets resulting from having to respond to an arbitrary infinitely long sequence of attacks at individual vertices. Each response permits moving each guard a distance of at most one, but must have one guard move to the attacked vertex. The smallest size of the sets for such a family is the eternal domination number. This has been referred to as the "all guards move model" or "eternal m-security" [7] and as "m-eternal domination" [4], and is one of a number of variations of problems involving mobile guards [8]. This paper builds on previous results on the eternal domination number for members of the family of $3 \times n$ grid graphs [2, 3, 4, 9], determining the outstanding values.

The domination number for all grid graphs, denoted $\gamma(G)$, has been determined [5, 6], with the value for $3 \times n$ grid graphs being $\gamma(P_3 \square P_n) = \lceil \frac{3n+1}{4} \rceil$, making this a lower bound for the eternal domination number of such graphs, which we denote $\gamma_{all}^{\infty}(P_3 \square P_n)$. Goldwasser, Klostermeyer, and Mynhardt [4] found that

$$\gamma_{all}^{\infty}(P_3 \square P_n) \leq \left\lceil \frac{8n}{9} \right\rceil \text{ for } n \geq 9$$

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and conjectured that $\gamma_{all}^{\infty}(P_3 \,\square\, P_n) = \lceil \frac{4n+5}{5} \rceil$ for $n > 9$. Finbow, Messinger, and van Bommel [2] disproved the conjecture and provided a tightening of the bounds to

$$\left\lceil \frac{4n+6}{5} \right\rceil \leq \gamma_{all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{6n+2}{7} \right\rceil \text{ for } n \geq 11.$$

Messinger and Delaney [9] developed a set of configurations for eternal dominating families which helped close the gap of the bounds to

$$\left\lceil \frac{4n+6}{5} \right\rceil \leq \gamma_{all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{4n+10}{5} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 0, 1, 3 \pmod 5 \\ 0 & \text{otherwise.} \end{cases}$$

This paper employs a variation of their configurations, as well as a proof that the lower bound is not tight, to remove the gap completely, and establish the value $\gamma_{all}^{\infty}(P_3 \,\square\, P_n) = \lceil \frac{4n+7}{5} \rceil$ for $n \geq 26$.

2 Definitions

Let $G = (V, E)$ be a graph. A *dominating set* of G is a subset of V whose closed neighbourhood is V. The smallest cardinality of a dominating set is denoted $\gamma(G)$ and is called the *domination number* of G. Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. Let $D, D' \in \mathbb{D}_q(G)$. We will say D can be *transformed* to D' (or D *transforms* to D') if $D = \{v_1, v_2, \ldots, v_q\}$, $D' = \{u_1, u_2, \ldots, u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \ldots, q$.

In the "eternal dominating set problem", a defender is given q guards to protect the graph from a series of attacks on vertices made by an attacker. An *eternal dominating family of G* is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and D transforms to D'. When the value of q in the above definition is known we will refer to this family as an eternal dominating family with q guards. For a graph G, the minimum value of q such that there exists an eternal dominating family with q guards is denoted $\gamma_{all}^{\infty}(G)$. A set $D \in \mathbb{D}_q(G)$ is an *eternal dominating set* or a *q-eternal dominating set* if it is a member of some eternal dominating family. Note that the set of all eternal dominating sets of a particular cardinality is an eternal dominating family, provided the family is non-empty.

The *Cartesian product* of graphs G and H is denoted by $G \,\square\, H$. The vertex set of $G \,\square\, H$ is $V(G \,\square\, H) = \{(u, v) | u \in V(G), v \in V(H)\}$, and two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. When $G = P_m$ and $H = P_n$, these graphs are also known as a *grids* or *grid graphs* of dimensions $m \times n$. The vertices of P_m (respectively P_n) are labeled in their usual ordering $u_1, u_2, \ldots u_m$ (resp. $v_1, v_2, \ldots v_n$). In this paper, we discuss the eternal domination numbers of grid graphs with $m = 3$. Each copy of P_3, corresponding to a vertex of P_n, is referred to as a column. We refer to each of the columns as the first column, second column, etc. and as column 1, column 2, etc. starting from one the columns corresponding to a leaf of P_n and proceeding consecutively.

In constructing eternal dominating families we make use of the symmetries of the $3 \times n$ grid graph. Given a dominating set $D \in \mathbb{D}_q(P_3 \,\square\, P_n)$, a vertical reflection of D (about the horizontal line of symmetry) is denoted D_v, while a horizontal reflection (about the vertical line of symmetry) is denoted D_h. A rotation of a dominating set D by 180° (which is the same as both the vertical reflection of D_h and the horizontal reflection of D_v) is denoted D_r. When we wish to discuss an arbitrary symmetry of a dominating set D, we denote it D_s.

3 Previous Results and Extensions

We begin with several observations of Beaton et. al [1] and Finbow et. al [2], and extend two of these results. We note that, by symmetry, statements and arguments referring to the first i columns also apply to the last i columns, for any i.

Theorem 1 *[1] Given dominating sets $D, E \in \mathbb{D}_q(P_m \Box P_n)$ and any arbitrary symmetry s resulting from a reflection or rotation, D transforms to E if and only if D_s transforms to E_s.*

Corollary 2 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$. Then the family*

$$\mathcal{F} = \mathcal{E} \cup \{D_h | D \in \mathcal{E}\} \cup \{D_v | D \in \mathcal{E}\} \cup \{D_r | D \in \mathcal{E}\}$$

is an eternal dominating family of $P_3 \Box P_n$.

Proof: Let $F \in \mathcal{F}$ be some dominating set in $\mathcal{F}$. If $F \in \mathcal{E}$, then since $\mathcal{E}$ is an eternal dominating family, for every possible attack $v \in V(P_3 \Box P_n)$, there exists a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and F transforms to D'.

Otherwise, if $F \notin \mathcal{E}$, there must exist a dominating set $D \in \mathcal{E}$ and some symmetry s of D such that $F = D_s$. Consider an attack on some $v \in V(P_3 \Box P_n)$. Let $v_s \in V(P_3 \Box P_n)$ be the image of v under the symmetry s. Since $\mathcal{E}$ is an eternal dominating family, there exists a dominating set $D' \in \mathcal{E}$ so that $v_s \in D'$ and D transforms to D'. It follows that $v \in D'_s$, the symmetry of D', and $F = D_s$ transforms to D'_s. As $D'_s \in \mathcal{F}$, and F was an arbitrary member of $\mathcal{F}$, it follows that $\mathcal{F}$ is an eternal dominating family of $P_3 \Box P_n$. ∎

Lemma 3 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$. If there are at least k guards in the first i columns for each dominating set $D \in \mathcal{E}$, then for any set $D' \in \mathcal{E}$ all of the following hold.*

1. *If there are at most k guards in the first $i+1$ columns, then there are k guards in the first i columns, no guards in column $i+1$ and three guards in column $i+2$.*
2. *If there are at most $k+1$ guards in the first $i+2$ columns, then there are $k+1$ guards in the first $i+1$ columns, no guards in column $i+2$ and at least two guards in column $i+3$.*
3. *If there are at most $k+2$ guards in the first $i+3$ columns, then there are $k+2$ guards in the first $i+2$ columns, no guards in column $i+3$ and at least two guards in column $i+4$.*
4. *If there are at most $k+3$ guards in the first $i+4$ columns, then there are $k+3$ guards in the first $i+3$ columns, no guards in column $i+4$ and at least one guard in column $i+5$.*
5. *There are at least $k+4$ guards in the first $i+5$ columns.*
6. *If there are at most $k+4$ guards in the first $i+6$ columns, then there are k guards in the first i columns, one guard in the middle of column $i+1$, no guards in column $i+2$, two guards (in the top row and bottom row) of column $i+3$, no guards in column $i+4$, one guard in the middle of column $i+5$, no guards in column $i+6$, and three guards in column $i+7$, as shown in Figure 1.*

1 · · · i	i+1	i+2	i+3	i+4	i+5	i+6	i+7	
			X				X	
k guards	X				X		X	
			X				X	

Figure 1: Only possible configuration for Lemma 3 (6.).

Proof: Items (1.) through (5.) were proven in [2]. We proceed to proving (6.).

Given the assumption of at most $k+4$ guards in the first $i+6$ columns, (5.) and (1.) show there are no guards in column $i+6$ and three guards in column $i+7$.

By assumption, there are at least k guards in the first i columns. Since there are no guards in column $i+6$, we require at least two guards in columns $i+3$ through $i+5$ to dominate the vertices of columns $i+4$ and $i+5$. This shows there are at most $k+2$ guards in the first $i+2$ columns.

Suppose there are exactly $k+2$ guards in the first $i+2$ columns. Then there are at most 2 guards in column $i+3$, column $i+4$ and column $i+5$. As D is a dominating set, it can be seen that there must be a guard in column $i+5$, a guard in column $i+4$ and 2 guard in column $i+2$. Hence there are k guards in the first $i+1$ columns. It follows from (1.) there are $k+3$ guards in the first $i+2$ columns, a contradiction showing there at most $k+1$ guards in the first $i+2$ columns.

By (2.), there are $k+1$ guards in the first $i+1$ columns, no guards in column $i+2$ and at least two guards in column $i+3$. As D is dominating, the remaining guard must be in the middle of column $i+5$, the two guards in column $i+3$ must be in the top and bottom row, and there is a guard in column $i+1$ which must be in the middle row. ∎

Corollary 4 ([2]) *In any eternal dominating set of $P_3 \square P_n$, for any $\ell \geq 2$, the first ℓ columns contain at least $\lceil \frac{4\ell-3}{5} \rceil$ guards.*

Lemma 5 ([2]) *Let $\mathcal{E}$ be a family of eternal dominating sets of $P_3 \square P_n$ and let $i \in \{0,1,2,3,4\}$. For every $D \in \mathcal{E}$, there are at least i guards in the first $i+1$ columns.*

4 Improving the Lower Bound

In this section, we establish an improved lower bound in the case $n \equiv 1 \pmod 5$. We note when $n \equiv 1 \pmod 5$, the lower bound $\lceil \frac{4n+6}{5} \rceil = \frac{4n+6}{5}$.

Corollary 6 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$. If there are at least 10 guards in the first 12 columns for each dominating set $D \in \mathcal{E}$, then for any $\ell \geq 3$, $\ell \neq 7$, the first ℓ columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards.*

Proof: For $3 \leq l \leq 11$, $\lceil \frac{4\ell-3}{5} \rceil = \lceil \frac{4\ell-2}{5} \rceil$ unless $l = 7$. For $l = 12$, $\lceil \frac{4\ell-2}{5} \rceil = 10$. So by assumption and Corollary 4, the first l columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards for $3 \leq \ell \leq 12$, unless $\ell = 7$. Further, by Lemma 3 (5.), if the result holds for $\ell = k$, then the result holds for $\ell = k+5$. By induction, for any $\ell \geq 8$, the first ℓ columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards. ∎

Lemma 7 *Let $n \equiv 1 \pmod 5$, $n \geq 26$. Let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$ with at most $\frac{4n+6}{5}$ guards and with the property that if $D \in \mathcal{E}$, then $D_r \in \mathcal{E}$. There is an eternal dominating set in $\mathcal{E}$ with nine guards in the first twelve columns.*

Proof: By Corollary 4, there must be at least 9 guards in the first 12 columns. Suppose then, by way of contradiction, each dominating set in $\mathcal{E}$ has at least 10 guards in the first 12 columns. Hence, since $D \in \mathcal{E}$ implies $D_r \in \mathcal{E}$, each dominating set in $\mathcal{E}$ has at least 10 guards in the last 12 columns.

Let l be the largest integer such that for any $k \leq l$, every eternal dominating set in $\mathcal{E}$ has at least k guards in the first $k+2$ columns, but there is a set $D \in \mathcal{E}$ with at most l guards in the first $l+3$ columns. By assumption and Lemma 4, $l \geq 10$. We claim, $l \leq n-9$. Suppose an eternal dominating set has $n-8$ guards in the first $n-6$ columns, by Lemma 4, there are at least five guards in the last 6 columns and hence the number of guards is at least $n-8+5 \leq \frac{4n+6}{5}$. This implies $n \leq 21$, a contradiction and therefore $l \leq n-9$. Recall there is a dominating set D which has at most l guards in the first $l+3$ columns. By Lemma 3 (1.), D has l guards in the first $l+2$ columns, no guards in column $l+3$ and 3 guards in column $l+4$. We consider two cases.

Case 1: $n-(l+4) = 7$. By Lemma 4 D has at least 5 guards in the remaining 7 columns. Therefore $\frac{4n+6}{5} \geq |D| \geq l+3+5 = (l+4)+4 = (n-7)+4 = n-3$. This implies $n \leq 21$, a contradiction.

Case 2: $n-(l+4) \neq 7$. Since $l \leq n-9$, $n-(l+4) \geq 5$. By Lemma 6 D has at least $\left\lceil \frac{4(n-(l+4))-2}{5} \right\rceil$ guards in the remaining $n-(l+4)$ columns. Hence,

$$\frac{4n+6}{5} \geq |D| \geq l+0+3+\left\lceil \frac{4(n-(l+4))-2}{5} \right\rceil.$$

Rearranging we obtain

$$\frac{4n+6}{5} \geq \frac{4n+6}{5} + \left\lceil \frac{l-9}{5} \right\rceil$$

which is false as, $l \geq 10$.

Both cases lead to a contradiction, establishing the result. ∎

Lemma 8 *For any $n \equiv 1 \pmod 5$, $n \geq 26$ $\gamma^{\infty}_{all}(P_3 \Box P_n) \geq \dfrac{4n+11}{5}$.*

Proof: Let $n \equiv 1 \pmod 5$, $n \geq 26$ be given and let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$ which uses $\frac{4n+6}{5}$ guards. Set $d = \frac{4n+6}{5}$ and let

$$\mathcal{E}' = \mathcal{E} \cup \{D_h | D \in \mathcal{E}\} \cup \{D_v | D \in \mathcal{E}\} \cup \{D_r | D \in \mathcal{E}\}.$$

By Corollary 2, $\mathcal{E}'$ is an eternal dominating family of $P_3 \Box P_n$. It follows from Lemma 7 that there exists an eternal dominating set $D \in \mathcal{E}'$ with 9 guards in the first 12 columns. By Corollary 4, any eternal dominating set of $P_3 \Box P_n$ must contain at least 9 guards in the first 11 columns, and hence by Lemma 3 (6.), D has 5 guards in the first 6 columns, one guard in the middle of column 7, no guards in column 8, two guards (in the top row and bottom row) of

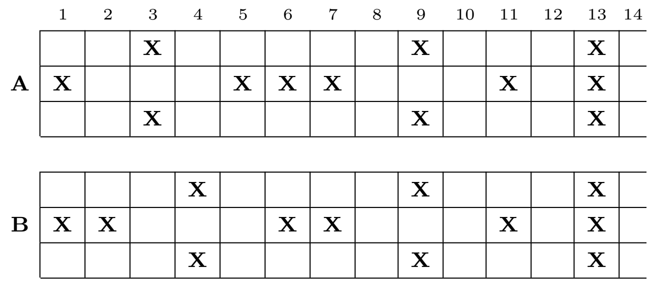

Figure 2: The two possible configurations for Lemma 8.

column 9, no guards in column 10, one guard in the middle of column 11, no guards in column 12 and three guards in column 13. Exhaustive search reveals there are two possibilities for the guards locations in the first 13 columns, as illustrated in Figure 2.

We wish to show that **A** is not part of a d-eternal dominating set. To establish this, we consider a sequence of two attacks and the corresponding transformations, as depicted in Figure 3. Specifically, the first attack is at the middle vertex of column 2. To successfully defend against the attack, the defender must move the guards to transform an eternal dominating set containing **A** to a d-eternal dominating set D' containing the attacked vertex (and thus also containing $\mathbf{A}'$). In particular, the defender must:

- move the guard in column 1 to the attacked vertex.
- move the guards in column 3 to column 2 so that D' has a vertex in the neighbourhood of each of the vertices of the first column.
- move the guards in the middle vertices of columns 5 through 7 one column to the left so that D' has guards in the neighbourhood of all the vertices in columns 4 through 6.
- move the guards in the top and bottom vertices in column 9 to column 8 so that D' has guards in the neighbourhood of all the vertices in columns 7 and 8.
- move the guard in the middle vertex of column 11 to the middle vertex of column 10 so that D' has guards in the neighbourhood of all the vertices in columns 9 and 10.
- move the guards in the top and bottom vertices in column 13 to column 12 so that D' has guards in the neighbourhood of all the vertices in column 11.

According to Corollary 4, the last $n-12$ columns of D' contain at least $\left\lceil \frac{4(n-12)-3}{5} \right\rceil$ guards, or $d-11$ guards as $n \equiv 1 \pmod 5$. This implies there are no other guards in the first 12 columns of D'.

With guards located on the vertices of D' (containing $\mathbf{A}'$), consider an attack on the middle vertex of column 9. To successfully defend against the attack, the defender must transform D' to a d-eternal dominating set D'' containing the attacked vertex (and thus $\mathbf{A}''$). The defender must move the guard in column 10 to the attacked vertex, and it is clear that in D'' there are 9 guards in the first 9 columns. Hence, there are $d-9$ guards in the remaining $n-9$ columns.

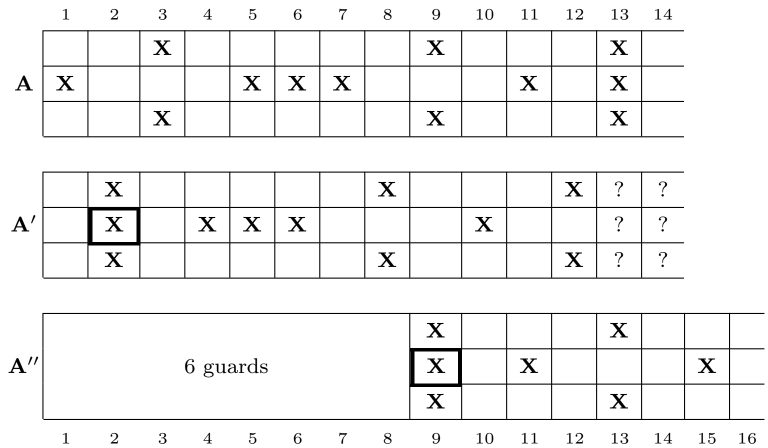

Figure 3: Two attacks and defender responses, starting from **A**.

By Corollary 4, every dominating set in $\mathcal{E}'$ has at least $\left\lceil \frac{4(n-15)-3}{5} \right\rceil$ guards in the last $n-15$ columns, or $d-13$ as $n \equiv 1 \pmod 5$.

With these two values ($i = n-15$ and $k = d-13$), Lemma 3 (6.) implies D'' has $d-13$ guards in the last $n-15$ columns, one guard in the middle of column 15, no guards in column 14, two guards (in the top and bottom row) of column 13, no guards in column 12, one guards in the middle row of column 11, no guards in column 10, and three guards in column 9. However, after moving the guard in column 10 to column 9 in response to the attack, no guard can be moved to the middle of column 11. Thus D' cannot transform to a d-eternal dominating set containing $\mathbf{A}''$, so $\mathbf{A}$ is not part of an eternal dominating set.

An almost identical argument can be used to establish $\mathbf{B}$ is not part of an eternal dominating set by considering an attack in the middle of the third column, followed by an attack in the middle of the ninth column. Therefore D does not exist. ■

Corollary 9 *For all $n \geq 26$,*

$$\gamma_{all}^{\infty}(P_3 \square P_n) \geq \left\lceil \frac{4n+7}{5} \right\rceil.$$

5 Arrangement for $n \equiv 2 \pmod 5$

Consider the two guard arrangements as illustrated in Figure 4.

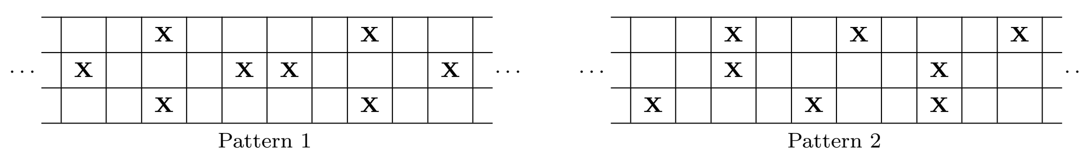

Figure 4: Two basic guard patterns.

Shifting these repeating patterns within the $3 \times n$ grid graph where $n \equiv 2 \pmod 5$ and providing an adjustment for the first and last columns can lead to several possible configurations. Specific ones are illustrated in Figure 5, where the ellipses in each configuration represent a continuation of the pattern from the center block in five-column block increments. Note that in the patterns for G, O, and P, the vertical reflection of the last three columns might be required, depending on the number of repetitions. Configuration B (Blue) follows Pattern 1 where the columns with two guards are those with number $\equiv 2 \pmod 5$, and an extra guard is placed in the middle of column 1. Y (Yellow) also follows Pattern 1 but is shifted one column to the right, and extra guards are placed in the middle of column 2 and the last column. R (Red) follows Pattern 1 shifted right again, and an extra guard is placed in the bottom of the last column. G (Green) follows Pattern 2 where the columns with two guards are are those with number $\equiv 2 \pmod 5$, and an extra guard is placed in the top of column 1. O (Orange) also follows Pattern 2 but is shifted one column to the right, and extra guards are placed in the middle of the first and last columns. Finally, P (Purple) follows Pattern 2 shifted right again, and an extra guard is placed in the middle of the last column.

Consider the B configuration. An attack on an unguarded vertex that appears in the same column as some guard can be defended by a move of all guards either up or down within each of their columns, leading to the G configuration (or the vertical reflection of G). This is shown in Figure 5, where a G in each of these positions of the B configuration indicates an attack on one of these vertices can be defended by a transition to the G configuration (or the vertical reflection of G). The vertices labeled O indicate an attack on one of these vertices can be defended by a transition to the O configuration (or the vertical reflection of O). Finally, the vertices labeled $\overline{\mathrm{G}}$ indicates an attack on one of these vertices can be defended by a transition to the horizontal reflection of the G configuration (or the vertical reflection of G). In what follows, when we refer to a configuration or pattern, we assume the vertical reflection if necessary.

As indicated in Figure 5, the Y configuration can transition to G, O, R, or $\overline{\mathrm{P}}$, the horizontal reflection of P, in response to an attack. The G configuration can transition to B, Y, O, $\overline{\mathrm{B}}$, the horizontal reflection of B, or the vertical reflection of G itself, indicated by the bullets ($\bullet$). The O configuration can transition to B, Y, G, $\overline{\mathrm{R}}$, the horizontal reflection of R, or the vertical reflection of O itself.

Consider the P configuration. Most of the possible attacks can be defended by transitions to R, O, horizontal reflections of O and Y, and the vertical reflection of P itself. However, consider an attack on the top vertex of the third column. The response can lead to the pattern of configuration xO illustrated in Figure 6. This is equivalent to configuration O except the guard at the top of the first column is instead at the top of the third column, thus creating a block with three guards in one column, which we call x.

An attack on the P configuration at one of the positions labeled RxO in the repeating pattern must also be defended by moving to a similar configuration where the guard defending the attack is in the middle of a column of three guards. A set of possible configurations is illustrated by the configuration RxO in Figure 6, which starts with the pattern of R, followed by the x pattern, and ends with the O pattern. The ellipses in this configuration represent zero or more repetitions of the pattern of the closest five columns from the center block. In this manner, an attack on the bottom of the eighth column of configuration P would lead to a configuration containing the first four columns of R, the five columns of the x block, followed by the necessary number of columns from the end of the O pattern.

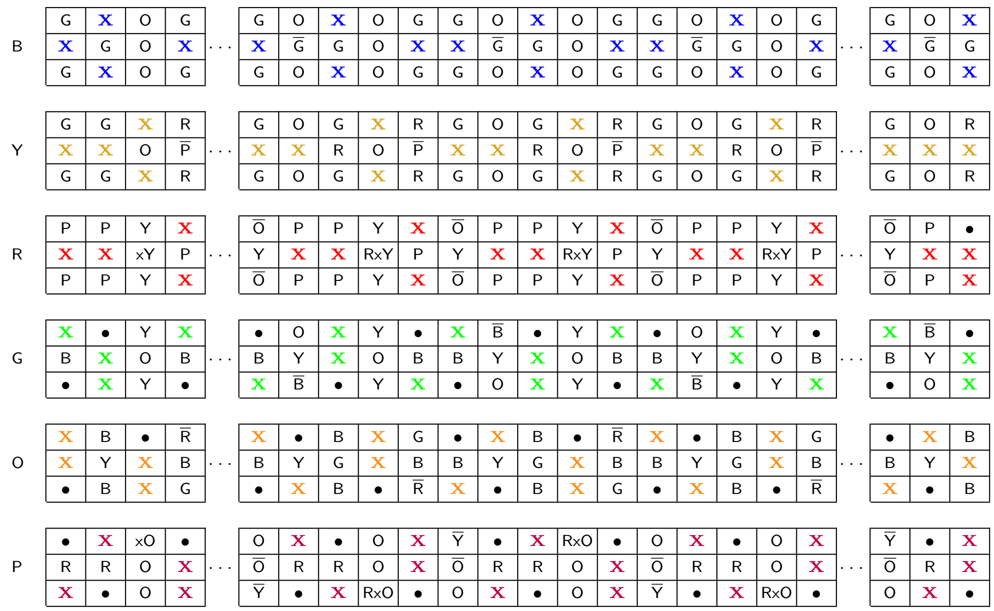

Figure 5: Six monochromatic guard configurations when $n \equiv 2 \pmod 5$.

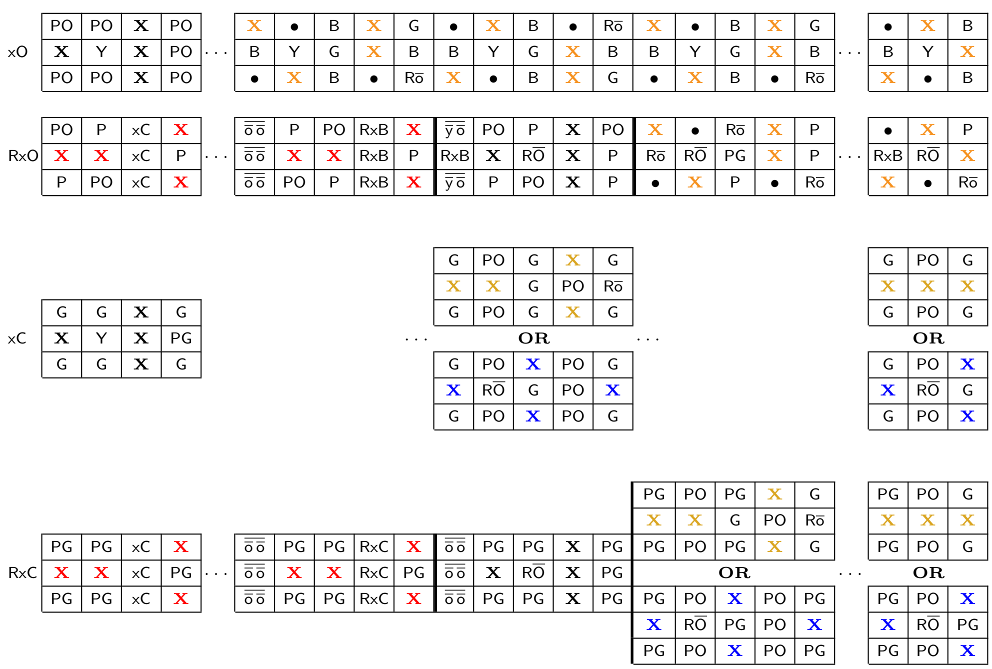

Figure 6: Configurations with x Blocks for $n \equiv 2 \pmod 5$.

Consider the R configuration. Most of the possible attacks can be defended by transitions to Y, P, or $\overline{\text{O}}$, the horizontal reflection of O. However, consider an attack on the middle vertex of the third column. The response can lead to the pattern of xY, which consists of the first four columns of the xC pattern illustrated in Figure 6, followed by the necessary number of Y blocks, and ending with the end of the Y configuration. This pattern is equivalent to that of configuration Y except one guard is shifted from the middle of the second column to the middle of the third column. In Figure 6, the configuration is labeled xC because the C represents any combination of Y and B blocks in the middle of the configuration, and any choice of ending from Y or B.

An attack on the R configuration at one of the positions labeled RxY in the repeating pattern must also be defended by moving to a similar configuration where the guard defending the attack is in the middle of a column of three guards. A set of possible configurations is illustrated by the configuration RxC in Figure 6 where only Y patterns are chosen, thus it starts with the pattern of R, followed by the x pattern, and ends with the Y pattern.

As shown, defending attacks on the configurations illustrated in Figure 6 require the six additional configurations illustrated in Figure 7. Configurations PO and PG start with the pattern of P and end with the pattern of O or G, respectively; that is, they contain the first four columns of P, followed by zero or more five-column blocks of the pattern of P, and end with the pattern of O or G. Configuration R$\overline{\text{O}}$ starts with the pattern of R, continues with the horizontal reflection of the middle blocks of the O pattern, and ends with a distinctive ending pattern.

Configuration R$\bar{\text{o}}$ also starts with the pattern of R, but this is followed by the horizontal reflection of the O pattern, starting with its specific ending, and continuing for as many columns as necessary. The $\bar{\text{y}}\,\bar{\text{o}}$ and $\bar{\text{o}}\,\bar{\text{o}}$ configurations start with the horizontal reflection of the the Y and O patterns, respectively, starting with their specific endings, for some number

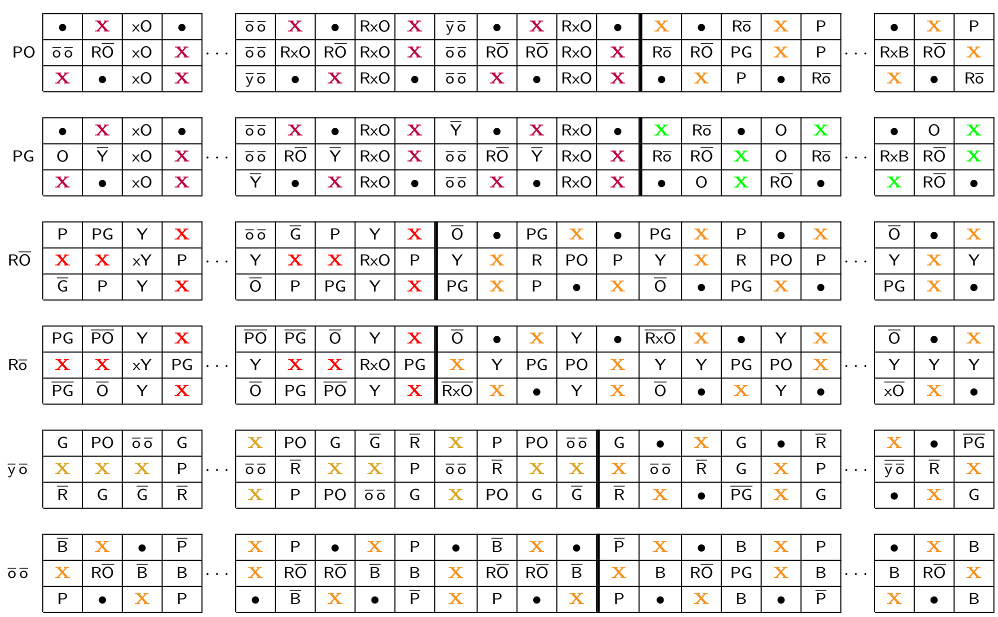

Figure 7: Remaining Configurations for $n \equiv 2 \pmod 5$.

of columns i, where $i \equiv 4 \pmod 5$. In either case, this pattern is followed by the horizontal reflection of the O pattern, starting with its specific ending, for the appropriate number of columns.

Lemma 10 *For $n \equiv 2 \pmod 5$, $n \geq 17$, $\gamma_{all}^{\infty}(P_3 \square P_n) \leq \frac{4n+7}{5}$.*

Proof: The configurations illustrated in Figures 5, 6, and 7 demonstrate how to construct one such eternal dominating family for any such possible value of n. Also included is one possible transformation for any attack on any of the dominating sets. ■

6 Arrangement for $n \equiv 3 \pmod 5$

In this section, we construct an eternal dominating family of $P_3 \square P_n$ when $n \equiv 3 \pmod 5$ by building on the configurations in Figures 5, 6, and 7. The basic idea is to create dominating sets by adding one column with one guard, either at the front or end of these configurations, and show the result is an eternal dominating family.

Lemma 11 *For any $n \equiv 3 \pmod 5$ with $n \geq 28$, $\gamma_{all}^{\infty}(P_3 \square P_n) \leq \dfrac{4n+8}{5}$.*

Proof: Let $n \equiv 3 \pmod 5$ be given and let $\mathcal{E}$ be the eternal dominating family presented in Section 5 for a $P_3 \square P_{n-1}$ grid. We form $\mathcal{F}'$ from $\mathcal{E}$ as follows.

1. For each configuration $D \in \mathcal{E}$, add to $\mathcal{F}'$ the configuration with a guard in the middle of the first column and which is identical to D on the remaining $n-1$ columns. The configuration created by this process (and its vertical reflection) will be denoted mD. We note that m$\overline{D}$ is the horizontal reflection of the configuration which is identical to D in the first $n-1$ columns with a guard in the middle of the last column. Therefore we will also denote $m\overline{D}$ with the notation Dm.

2. For each configuration $D \in \mathcal{E}$ with a guard in the bottom row (respectively in the top row) of the first column, add to $\mathcal{F}'$ the configuration of $P_3 \square P_n$ which is identical to D on the last $n-1$ columns and with a guard in the top row (respectively bottom row) of the new first column. A configuration of this form (and its vertical reflection) will be denoted tD, and t$\overline{D}$ will also be denoted by Dt. In the three cases where configurations in $\mathcal{E}$ have a guard in both the top row and the bottom row - Bt, RxCt and xCt - the two sets that may be formed with this process are vertical reflections of each other.

The sets that have been added to $\mathcal{F}'$ associated with the configurations B and $\overline{B}$ are shown in Figure 8.

Let $\mathcal{F} = \mathcal{F}' \cup \{D_h | D \in \mathcal{F}'\} \cup \{D_v | D \in \mathcal{F}'\} \cup \{D_r | D \in \mathcal{F}'\}$. To establish the Lemma, it suffices to show the family $\mathcal{F}$ is an eternal dominating family of $P_3 \square P_n$.

From the definition above, each set in $\mathcal{F}$ is a dominating set with $\frac{4n+8}{5}$ guards. Let $D \in \mathcal{F}'$. We note there exists a set $E \in \mathcal{E}$ such that D is either mE or tE. Consider an attack on a vertex $v \in V(P_3 \square P_n) - D$.

We have four cases.

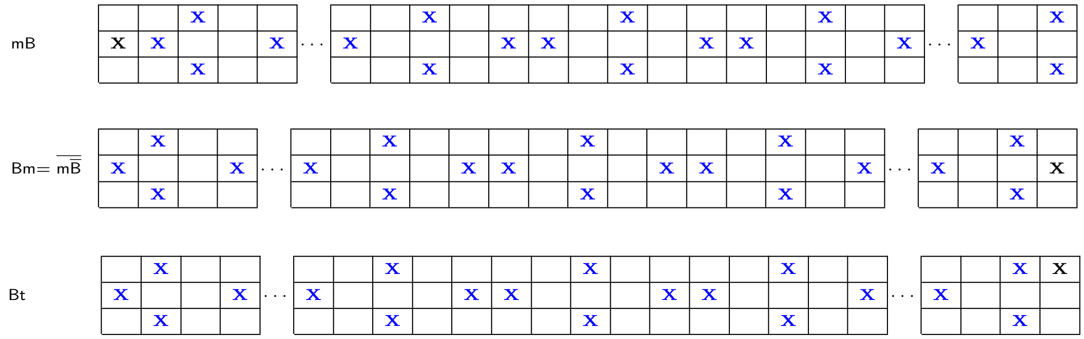

Figure 8: New configurations formed from adding one column to B or $\overline{B}$.

Case 1: v is not in the first column.

The defender considers only the position of the guards and the attacker on the last $n-1$ columns. The guards on these vertices are positioned on the set E. As $\mathcal{E}$ is an eternal dominating family for a $P_3 \Box P_{n-1}$ grid, there is a set $E' \in \mathcal{E}$ so that $v \in E'$ and E transforms to E'. It follows that D transforms to $D' = mE'$ and $v \in D' \in \mathcal{F}$.

Case 2: v is the middle row of the first column.

Clearly $v \notin D$, so $D = tE$. The set D transforms to $D' = mE$ and $v \in D' \in \mathcal{F}$.

Case 3: v is the top row (or bottom row) of the first column and $D = mE$.

Let u be the vertex in the bottom row of the second column. If $u \in D$, then D transforms to tD and $v \in tD$. If $u \notin D$, the defender momentarily considers only the position of the guards and the attacker on the last $n-1$ columns. The guards on these vertices are positioned on the set E. As $\mathcal{E}$ be the eternal dominating family for a $P_3 \Box P_{n-1}$ grid, there is a set $E' \in \mathcal{E}$ so that $u \in E'$ and E transforms to E'. It follows that D transforms to $D' = tE'$ and $u, v \in D' \in \mathcal{F}$.

Case 4: v is the top row (or bottom row) of the first column and $D = tE$.

Let u be the vertex in the bottom row of the second column. By definition of $\mathcal{F}'$, $u \in D$. Careful inspection shows D is one of the sets listed in Table 1. The second column represents a set $D' \in \mathcal{F}$ so that $v \in D'$ and D transforms to D'.

We conclude that for any set $D \in \mathcal{F}'$ and for any attack on a vertex $v \in V(P_3 \Box P_n) - D$, there exists a set $D' \in \mathcal{F}$ such that $v \in D'$ and D transforms to D'. If $D \in \mathcal{F} - \mathcal{F}'$, for some symmetry s, $D_s \in \mathcal{F}'$. Consider an attack on some $v \in V(P_3 \Box P_n) - D$. Let $v_s \in V(P_3 \Box P_n)$ be the image of v under the symmetry s. As $D_s \in \mathcal{F}'$, there exists a dominating set $D' \in \mathcal{F}$ so that $v_s \in D'$ and D_s transforms to D'. It follows that $v \in D'_s$, the symmetry of D', and D transforms to D'_s. As $D'_s \in \mathcal{F}$, it follows that $\mathcal{F}$ is an eternal dominating family of $P_3 \Box P_n$. ■

7 Main Result

We are now ready to present the main result of the paper.

Configuration	Response to attack
tG	tG
tO	tO
tP	$\overline{\text{mP}}$
Bt	Gt
Gt	Bt
Rt	Rt
Pt	Rt

Configuration	Response to attack
xCt	Gt
RxCt	PGt
PGt	RxCt
R$\overline{\text{O}}$t	R$\overline{\text{O}}$t
R$\bar{\text{o}}$t	mP
tPG	Pm
tPO	Pm

Table 1: Response to attacks from Case 4.

Theorem 12 *For all $n \geq 26$,*

$$\gamma^{\infty}_{all}(P_3 \square P_n) = \left\lceil \frac{4n+7}{5} \right\rceil.$$

Proof: By Corollary 9, Lemma 10 and Lemma 11, the result holds when $n \equiv 2, 3 \pmod 5$. In [4], it is established that for $2 \leq k \leq 5$, $\gamma^{\infty}_{all}(P_3 \square P_k) = k$. Noting the first m columns can be guarded independently of the last k columns, $2 \leq k \leq 5$,

$$\gamma^{\infty}_{all}(P_3 \square P_{m+k}) \leq \gamma^{\infty}_{all}(P_3 \square P_m) + \gamma^{\infty}_{all}(P_3 \square P_k) = \gamma^{\infty}_{all}(P_3 \square P_n) + k. \tag{1}$$

Note that $\gamma^{\infty}_{all}(P_3 \square P_{21}) = 18$ [2]. When $n = 26$, the result now follows from (1) and Corollary 9. For $n > 26$, $n \equiv 0, 1, 4 \pmod 5$ there exists integers m and k so that $m \equiv 2 \pmod 4$, $2 \leq k \leq 4$ and $n = m+k$. Therefore the result follows from (1), Corollary 9, and Lemma 10. ∎

With this result and previously determined values for smaller grids, the eternal domination numbers for all $3 \times n$ grid graphs are now determined, and can be summarized as:

$$\gamma^{\infty}_{all}(P_3 \square P_n) = \begin{cases} \left\lceil \frac{6n+2}{7} \right\rceil & \text{if } n \leq 11 \\ \left\lceil \frac{4n+6}{5} \right\rceil & \text{if } 11 < n \leq 22 \\ \left\lceil \frac{4n+7}{5} \right\rceil & \text{otherwise.} \end{cases}$$

As a sequence starting with the eternal domination number of a 3×1 grid graph, it is OEIS Sequence Number A289188 [10], presented as follows:

$$2, 2, 3, 4, 5, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14, 14, 15, 16, 17, 18, 18, 19,$$

$$20, 21, 22, 23, 23, 24, 25, 26, 27, 27, 28, 29, 30, 31, 31, 32, 33, 34, 35, 35, \ldots$$

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