# Calculus 112 Practice Problems

## Section 7.4 Problems #38, #45, #46, #48

**38.** The denominator $x^2 - 3x + 2$ can be factored as $(x-1)(x-2)$. Splitting the integrand into partial fractions with denominators $(x-1)$ and $(x-2)$, we have

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiplying by $(x-1)(x-2)$ gives the identity

$$x = A(x-2) + B(x-1)$$

so

$$x = (A+B)x - 2A - B.$$

Since this equation holds for all $x$, the constant terms on both sides must be equal. Similarly, the coefficient of $x$ on both sides must be equal. So

$$\begin{aligned} -2A - B &= 0 \\ A + B &= 1. \end{aligned}$$

Solving these equations gives $A = -1$, $B = 2$ and the integral becomes

$$\int \frac{x}{x^2-3x+2}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx = -\ln|x-1| + 2\ln|x-2| + C.$$

**45.** Since $x^2 + x^4 = x^2(1 + x^2)$ cannot be factored further, we write

$$\frac{x-2}{x^2+x^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}.$$

Multiplying by $x^2(1 + x^2)$ gives

$$x - 2 = Ax(1 + x^2) + B(1 + x^2) + (Cx + D)x^2$$
$$x - 2 = (A + C)x^3 + (B + D)x^2 + Ax + B,$$

so

$$A + C = 0$$
$$B + D = 0$$
$$A = 1$$
$$B = -2.$$

Thus, $A = 1$, $B = -2$, $C = -1$, $D = 2$, and we have

$$\int \frac{x-2}{x^2+x^4}\,dx = \int \left(\frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}\right) dx = \int \frac{dx}{x} - 2\int \frac{dx}{x^2} - \int \frac{x\,dx}{1+x^2} + 2\int \frac{dx}{1+x^2}$$
$$= \ln|x| + \frac{2}{x} - \frac{1}{2}\ln\left|1 + x^2\right| + 2\arctan x + K.$$

We use $K$ as the constant of integration, since we already used $C$ in the problem.

**46.** Let $x = 3\sin\theta$ so $dx = 3\cos\theta\,d\theta$, giving

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\theta\,d\theta = \int \frac{(9\sin^2\theta)(3\cos\theta)}{3\cos\theta}\,d\theta = 9\int \sin^2\theta\,d\theta.$$

Integrating by parts and using the identity $\cos^2\theta + \sin^2\theta = 1$ gives

$$\int \sin^2\theta\,d\theta = -\sin\theta\cos\theta + \int \cos^2\theta\,d\theta = -\sin\theta\cos\theta + \int (1 - \sin^2\theta)\,d\theta$$

$$\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.$$

Since $\sin\theta = x/3$ and $\cos\theta = \sqrt{1 - x^2/9} = \sqrt{9-x^2}/3$, and $\theta = \arcsin(x/3)$, we have

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = 9\int \sin^2\theta\,d\theta = -\frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + C$$
$$= -\frac{9}{2}\cdot\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C = -\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C$$

**48.** Let $t = \tan\theta$ so $dt = (1/\cos^2\theta)d\theta$. Since $\sqrt{1+\tan^2\theta} = 1/\cos\theta$, we have

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = \int \frac{1/\cos^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\cos\theta}{\tan^2\theta\cos^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

The last integral can be evaluated by guess-and-check or by substituting $w = \sin\theta$. The result is

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{\sin\theta} + C.$$

Since $t = \tan\theta$ and $1/\cos^2\theta = 1 + \tan^2\theta$, we have

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+t^2}}.$$

In addition, $\tan\theta = \sin\theta/\cos\theta$ so

$$\sin\theta = \tan\theta\cos\theta = \frac{t}{\sqrt{1+t^2}}.$$

Thus

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = -\frac{\sqrt{1+t^2}}{t} + C.$$