Calculus 112 Practice Problems

Section 9.4 Problems #14, #19, #24, #27, #29, #35-43

14. Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges.

$n=1$

19. Since $a_n = 2^n/(n^3+1)$, replacing n by $n+1$ gives $a_{n+1} = 2^{n+1}/((n+1)^3+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^3+1}}{\frac{2^n}{n^3+1}} = \frac{2^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{2^n} = 2\frac{n^3+1}{(n+1)^3+1},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2.$$

Since $L > 1$ the ratio test tells us that the series $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$ diverges.

24. Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

27. Let $a_n = 1/e^n$. Then replacing n by $n+1$ we have $a_{n+1} = 1/e^{n+1}$. Since $e^{n+1} > e^n$, we have $\frac{1}{e^{n+1}} < \frac{1}{e^n}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ converges by the alternating series test. We can also observe that the series is geometric with ratio $x = -1/e$ can hence converges since $|x| < 1$.

29. The series $\sum \frac{(-1)^n}{2n}$ converges by the alternating series test. However $\sum \frac{1}{2n}$ diverges because it is a multiple of the harmonic series. Thus $\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

35. We have

$$\frac{a_n}{b_n} = \frac{(5n+1)/(3n^2)}{1/n} = \frac{5n+1}{3n},$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n+1}{3n} = \frac{5}{3} = c \neq 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, the original series diverges.

36. We have

$$\frac{a_n}{b_n} = \frac{((1+n)/(3n))^n}{(1/3)^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n,$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = c \neq 0.$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, the original series converges.

37. The n^{th} term is $a_n = 1 - \cos(1/n)$ and we are taking $b_n = 1/n^2$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2}.$$

This limit is of the indeterminate form $0/0$ so we evaluate it using l'Hopital's rule. We have

$$\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2} = \lim_{n\to\infty} \frac{\sin(1/n)(-1/n^2)}{-2/n^3} = \lim_{n\to\infty} \frac{1}{2}\frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{1}{2}\frac{\sin x}{x} = \frac{1}{2}.$$

The limit comparison test applies with $c = 1/2$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore $\sum(1-\cos(1/n))$ also converges.

38. The n^{th} term $a_n = 1/(n^4 - 7)$ behaves like $1/n^4$ for large n, so we take $b_n = 1/n^4$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^4-7)}{1/n^4} = \lim_{n\to\infty} \frac{n^4}{n^4-7} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^4$ converges because $p = 4 > 1$. Therefore $\sum 1/(n^4 - 7)$ also converges.

39. The n^{th} term $a_n = (n+1)/(n^2+2)$ behaves like $n/n^2 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+1)/(n^2+2)}{1/n} = \lim_{n\to\infty} \frac{n^2+n}{n^2+2} = 1.$$

The limit comparison test applies with $c = 1$. Since the harmonic series $\sum 1/n$ diverges, the series $\sum(n+1)/(n^2+2)$ also diverges.

40. The n^{th} term $a_n = (n^3 - 2n^2 + n + 1)/(n^4 - 2)$ behaves like $n^3/n^4 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3-2n^2+n+1)/(n^4-2)}{1/n} = \lim_{n\to\infty} \frac{n^4-2n^3+n^2+n}{n^4-2} = 1.$$

The limit comparison test applies with $c = 1$. The harmonic series $\sum 1/n$ diverges. Thus $\sum \left(n^3 - 2n^2 + n + 1\right) / \left(n^4 - 2\right)$ also diverges.

41. The n^{th} term $a_n = 2^n/(3^n - 1)$ behaves like $2^n/3^n$ for large n, so we take $b_n = 2^n/3^n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(3^n-1)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{1}{1-3^{-n}} = 1.$$

The limit comparison test applies with $c = 1$. The geometric series $\sum 2^n/3^n = \sum(2/3)^n$ converges. Therefore $\sum 2^n/(3^n - 1)$ also converges.

42. The n^{th} term $a_n = 1/(2\sqrt{n} + \sqrt{n+2})$ behaves like $1/(3\sqrt{n})$ for large n, so we take $b_n = 1/(3\sqrt{n})$. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2\sqrt{n}+\sqrt{n+2})}{1/(3\sqrt{n})} &= \lim_{n\to\infty} \frac{3\sqrt{n}}{2\sqrt{n}+\sqrt{n+2}} \\ &= \lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}\left(2+\sqrt{1+2/n}\right)} \\ &= \lim_{n\to\infty} \frac{3}{2+\sqrt{1+2/n}} = \frac{3}{2+\sqrt{1+0}} \\ &= 1.\end{aligned}$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(3\sqrt{n})$ diverges because it is a multiple of a p-series with $p = 1/2 < 1$. Therefore $\sum 1/(2\sqrt{n} + \sqrt{n+2})$ also diverges.

43. The n^{th} term,

$$a_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{4n^2-2n},$$

behaves like $1/(4n^2)$ for large n, so we take $b_n = 1/(4n^2)$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(4n^2-2n)}{1/(4n^2)} = \lim_{n\to\infty} \frac{4n^2}{4n^2-2n} = \lim_{n\to\infty} \frac{1}{1-1/(2n)} = 1.$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(4n^2)$ converges because it is a multiple of a p-series with $p = 2 > 1$. Therefore $\sum \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$ also converges.