St. Francis Xavier University
Department of Computer Science

CSCI 356: Theory of Computing
Final Examination
December 11, 2021
2:00pm–4:30pm

Student Name: ______________________________

Email Address: ______________________________

Instructor: T. J. Smith (Section 10)

Format:

The exam is 150 minutes long. The exam consists of 7 questions worth a total of 75 marks. The exam booklet contains 10 pages, including the cover page and one blank page at the back of the exam booklet for rough work.

Reference Materials:

None.

Instructions:

1. Write your name and email address in the spaces above.
2. Answer each question either in the space provided or on a blank page. If you use a blank page to write your answer, indicate this clearly in the space provided for the question. Show all of your work.
3. Ensure that your exam booklet contains 10 pages. Do not detach any pages from your exam booklet.
4. Do not use any unauthorized reference materials or devices during this exam.
5. Sign in the space below. Your signature indicates that you understand and agree to these instructions and the university's examination policies.

Signature: ______________________________

Question	Marks	Score
1	20	
2	10	
3	5	
4	8	
5	10	
6	12	
7	10	
Total	75	

Multiple Choice

[20 marks] 1. For each of the following questions, select exactly one answer by circling the associated letter. Incorrect answers will not be penalized. Answers with more than one letter circled will be marked as incorrect.

(a) Let L be the language consisting of all words over $\Sigma = \{\texttt{b}, \texttt{c}\}$ having an equal number of bs and cs. The language L is denoted by the regular expression:

A. $(\texttt{b} + \texttt{c})^* + (\texttt{bb} + \texttt{cc})^*$
B. $(\texttt{bc} + \texttt{cb})^*$
C. $(\texttt{bc} + \texttt{cb} + \texttt{bbcc} + \texttt{bcbc} + \texttt{bccb} + \texttt{cbbc} + \texttt{cbcb} + \texttt{ccbb})^*$
D. $((\texttt{bc} + \texttt{cb})^* + (\texttt{bbcc} + \texttt{bcbc} + \texttt{bccb} + \texttt{cbbc} + \texttt{cbcb} + \texttt{ccbb})^*)^*$
E. None of the above.

(b) Let M and N be languages where $M \subseteq N$. Which of the following statements is **true**?

A. If M is context-free, then N is context-free.
B. If N is context-free, then M is context-free.
C. If M is non-context-free, then N is infinite.
D. If M is infinite, then N is non-context-free.

(c) Let $V = \{S\}$ and $\Sigma = \{\texttt{a}, \texttt{b}\}$. What language is generated by the following grammar?

$$S \to \texttt{a}S\texttt{bb} \mid \texttt{a}S\texttt{b} \mid \epsilon$$

A. $\{\texttt{a}^i\texttt{b}^{2k} \mid 0 \leq i \leq k\}$
B. $\{\texttt{a}^i\texttt{b}^k \mid 0 \leq i \leq k \leq 2i\}$
C. $\{\texttt{a}^i\texttt{b}^k \mid 0 \leq k \leq 2i\}$
D. None of the above.

(d) Let L be a regular language, let $M = \{\epsilon\}$, and let $N = \emptyset$. Let $\cdot$ denote the concatenation operation. Which of the following statements is **false**?

A. $L \cdot M = L$
B. $M \cdot N = M$
C. $N \cdot L = N$
D. $M \cdot M = \{\epsilon\}$
E. $N^* = \{\epsilon\}$

(e) Which of the following statements is **false**?

A. The class of languages recognized by deterministic pushdown automata is smaller than the class of languages recognized by nondeterministic pushdown automata.
B. Every regular language can be recognized by a Turing machine.
C. Every multi-tape Turing machine can be simulated by a single-tape Turing machine.
D. It is possible for a finite automaton to enter an infinite loop during its computation.
E. All of the above statements are true.

(f) Which of the following sets corresponds exactly to the class of decidable languages?

A. All languages recognized by nondeterministic pushdown automata.
B. All languages recognized by halting Turing machines.
C. All languages recognized by Turing machines that may or may not halt.
D. All languages that can be recognized in nondeterministic polynomial time.
E. All languages that have infinite size.

(g) Suppose we have two decision problems A and B, and we know that $A \leq_m B$ and A is undecidable. What can be said about B?

A. B is always decidable.
B. B may or may not be decidable depending on A.
C. B is always undecidable.
D. No such problems A and B can exist.
E. Not enough information is given.

(h) Let $\mathcal{M}$ be a Turing machine. What is the difference between A_{TM} and $HALT_{\mathsf{TM}}$?

A. A_{TM} applies only when $\mathcal{M}$ is nondeterministic, while $HALT_{\mathsf{TM}}$ applies only when $\mathcal{M}$ is deterministic.
B. A_{TM} asks whether $\mathcal{M}$ accepts an input word, while $HALT_{\mathsf{TM}}$ asks whether $\mathcal{M}$ does not loop forever on an input word.
C. A_{TM} is semidecidable, while $HALT_{\mathsf{TM}}$ is not semidecidable.
D. A_{TM} can be decided using the same procedure as A_{CFG}, while $HALT_{\mathsf{TM}}$ cannot be decided at all.

(i) What does it mean for a function $f(n)$ to be Big-O of another function $g(n)$?

A. The function $f(n)$ is always larger than the function $g(n)$.
B. For all large-enough values of n, some constant times $f(n)$ is bounded below by $g(n)$.
C. We can replace $g(n)$ with $f(n)$ in any expression to produce an equivalent expression.
D. For all large-enough values of n, $f(n)$ is bounded above by some constant times $g(n)$.

(j) Which of the following results is commonly known as the Cook–Levin theorem?

A. SATISFIABILITY is NP-complete.
B. $\mathsf{P} \subseteq \mathsf{NP}$.
C. A_{TM} is undecidable.
D. Americans and Soviets aren't so different after all.

Short Answer

[10 marks] 2. For each of the following questions, give a 1–2 sentence answer.

(a) What is the difference between a deterministic computation and a nondeterministic computation?

(b) Suppose we have a pushdown automaton with two stacks. What is the largest class of languages that this model can recognize, and why?

(c) What is the Church–Turing thesis? (Either state the thesis itself or describe what it tells us.)

(d) What is the difference between decidability and semidecidability?

(e) Let X be a complexity class. What does it mean for a decision problem to be X-complete? (Either describe completeness or give the conditions for a decision problem to be complete.)

[5 marks] 3. For each of the following pairs of language/complexity classes, write the relationship between each class in the provided blank space. You may select from the following relationships:

- Equal: $=$
- Non-strict inclusion: $\subseteq$ or $\supseteq$
- Strict inclusion: $\subset$ or $\supset$

For example, if you were given the pair "P _____ NP", the correct answer would be $\subseteq$ since we know that P is contained in NP, but we do not know whether the inclusion is strict or whether inclusion holds in the other direction.

Note that a strict inclusion $A \subset B$ indicates both that all elements of A are elements of B, and that there exists at least one element of B that is not also an element of A. If you write a non-strict inclusion when the inclusion is strict, or vice versa, then you will receive half marks for that question.

(a) REG _______________ CFL

(b) D _______________ $\mathsf{SD} \cap \mathsf{coSD}$

(c) D _______________ REG

(d) NP _______________ $\bigcup_{k \geq 0} \mathsf{DTIME}(n^k)$

(e) $\mathsf{DTIME}(n^2)$ _______________ $\mathsf{NTIME}(n^2)$

Long Answer

[8 marks] 4. Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. Consider the language

$$L = \{\mathtt{a}^i \mathtt{b}^k \mathtt{c}^{2i} \mathtt{d}^{3m} \mid i \geq k \geq 1, m \geq 1\}.$$

This language is non-context-free, and we can use the pumping lemma to prove non-context-freeness. Below, the first and last parts of the proof are given to you. Fill in the rest of the proof.

—

Assume by way of contradiction that L is context-free, and let p denote the pumping constant given by the pumping lemma. We choose the word $w =$ ______________________________. Clearly, $w \in L$ and $|w| \geq p$. Then, there exists a decomposition $w = uvxyz$ satisfying the three conditions of the pumping lemma.

We consider each case:

In all cases, one of the conditions of the pumping lemma is violated. As a consequence, the language cannot be context-free.

[10 marks] 5. Consider the Turing machine $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where $Q = \{q_0, q_1, q_{\text{accept}}, q_{\text{reject}}\}$, $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, $\Gamma = \Sigma \cup \{\sqcup\}$, and δ is defined as follows:

$$
\begin{aligned}
\delta(q_0, \mathtt{a}) &= (q_0, \mathtt{b}, R);\\
\delta(q_0, \mathtt{b}) &= (q_1, \mathtt{b}, R);\\
\delta(q_0, \sqcup) &= (q_{\text{accept}}, \sqcup, L);\\
\delta(q_1, \mathtt{a}) &= (q_{\text{reject}}, \mathtt{a}, L);\\
\delta(q_1, \mathtt{b}) &= (q_{\text{reject}}, \mathtt{b}, L);\\
\delta(q_1, \sqcup) &= (q_{\text{reject}}, \sqcup, L).
\end{aligned}
$$

(a) Give the sequence of configurations that $\mathcal{M}$ enters when it is given the input word `aaa`.

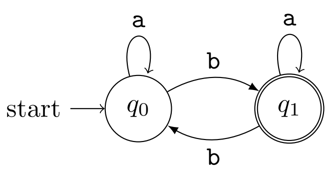
(b) Give the sequence of configurations that $\mathcal{M}$ enters when it is given the input word `aba`.

(c) What language does $\mathcal{M}$ recognize? Briefly justify your answer.

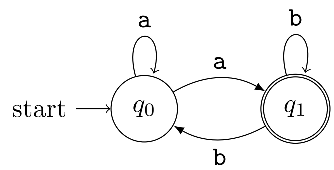
[12 marks] 6. (a) Recall the following decision problems:

$$A_{\mathsf{NFA}} = \{\langle \mathcal{B}, w\rangle \mid \mathcal{B} \text{ is a nondeterministic finite automaton that accepts input word } w\}$$
$$EQ_{\mathsf{NFA}} = \{\langle \mathcal{B}, \mathcal{C}\rangle \mid \mathcal{B} \text{ and } \mathcal{C} \text{ are both nondeterministic finite automata and } L(\mathcal{B}) = L(\mathcal{C})\}$$

Consider the following nondeterministic finite automata:



NFA $\mathcal{B}$



NFA $\mathcal{C}$

For each of the following questions, answer "yes" or "no" and justify your answer.

i. Is $\langle \mathcal{B}, \texttt{bab}\rangle \in A_{\mathsf{NFA}}$?

ii. Is $\langle \mathcal{C}, \texttt{abba}\rangle \in A_{\mathsf{NFA}}$?

iii. Is $\langle \mathcal{B}, \mathcal{C}\rangle \in EQ_{\mathsf{NFA}}$?

(b) Consider the following decision problem:

$$U_{\mathsf{NFA}} = \{\langle \mathcal{B}\rangle \mid \mathcal{B} \text{ is a nondeterministic finite automaton and } L(\mathcal{B}) = \Sigma^*\}.$$

Prove that U_{NFA} is decidable by giving a decision algorithm.

[10 marks] 7. (a) Consider the following decision problem:

> <u>TRIANGLES</u>
> Given: an undirected graph $G = (V, E)$
> Determine: whether G contains a triangle; i.e., a triple of vertices u, v, and w that are connected to each other by edges $\{u, v\}$, $\{v, w\}$, and $\{w, u\}$

Show that TRIANGLES is in P.

Hint. You may find the following fact useful: a graph with m vertices contains $\binom{m}{3} \in O(m^3)$ triples of vertices.

(b) Explain how, if $\mathsf{P} = \mathsf{NP}$, we would be able to determine the existence of a satisfying assignment to a given Boolean formula ϕ in polynomial time.

[2 marks] *Bonus.* What was your favourite part of this course, and why?

This blank page may be used for rough work.