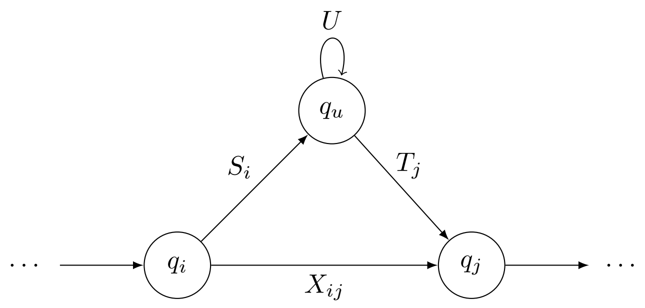
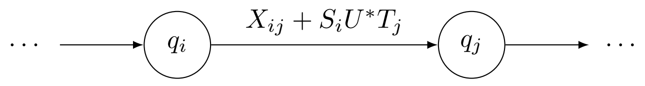
Now, we eliminate all states q_u of $\mathcal{M}$ that are neither initial nor accepting. Suppose that $\mathcal{M}$ contains the following substructure:



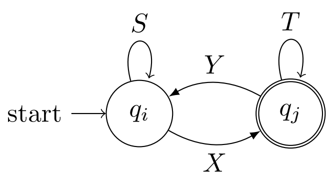
In this substructure, all transitions from states $q_i \neq q_u$ to state q_u are labelled by a regular expression S_i; all transitions from state q_u to states $q_j \neq q_u$ are labelled by a regular expression T_j, and for all such states q_i and q_j the transition between these states is labelled by a regular expression X_{ij}, or $\emptyset$ if no such transition exists. Lastly, any loop from q_u to itself is labelled by a regular expression U, or $\emptyset$ if no loop exists.

We may eliminate state q_u from $\mathcal{M}$ as follows: for each pair of states q_i and q_j, the regular expression X_{ij} on the transition is replaced by $X_{ij} + S_i U^* T_j$.



We then repeat this procedure for all non-initial and non-accepting states until the only states remaining in the finite automaton are the single initial and accepting states.

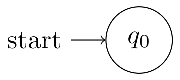
Suppose that, at this stage of the algorithm, our finite automaton is of the following form, where S, T, X, and Y are regular expressions:



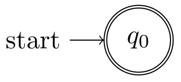
If any of these transitions do not exist, then we simply add them to the finite automaton labelled by $\emptyset$. Then the language recognized by $\mathcal{M}$ is represented by the regular expression $S^* X (T + Y S^* X)^*$.

($\Leftarrow$): To prove this direction of the statement, we will convert a regular expression $\boldsymbol{r}$ to a nondeterministic finite automaton $\mathcal{M}$ using a construction known as the *McNaughton–Yamada–Thompson algorithm.* We consider each of the "base" regular expressions:

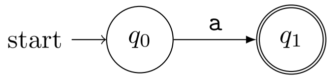
1. If $\boldsymbol{r} = \emptyset$, then $L(\boldsymbol{r}) = \emptyset$ and this language is recognized by the following nondeterministic finite automaton:



2. If $\boldsymbol{r} = \epsilon$, then $L(\boldsymbol{r}) = \{\epsilon\}$ and this language is recognized by the following nondeterministic finite automaton:



3. If $\boldsymbol{r} = a$ for some $a \in \Sigma$, then $L(\boldsymbol{r}) = \{a\}$ and this language is recognized by the following nondeterministic finite automaton:

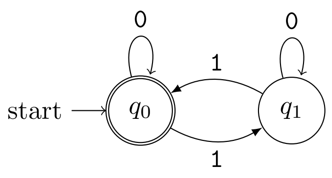


4. If $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2$ for some regular expressions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, then the corresponding language is recognized by the nondeterministic finite automaton constructed in the proof of Theorem 21.
5. If $\boldsymbol{r} = \boldsymbol{r}_1\boldsymbol{r}_2$ for some regular expressions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, then the corresponding language is recognized by the nondeterministic finite automaton constructed in the proof of Theorem 22.
6. If $\boldsymbol{r} = \boldsymbol{r}^*$ for some regular expression $\boldsymbol{r}$, then the corresponding language is recognized by the nondeterministic finite automaton constructed in the proof of Theorem 23.

Since, in each case, we can convert the "base" regular expression to a nondeterministic finite automaton, and we can then determinize the overall finite automaton using our procedure from Theorem 19, we know by Theorem 24 that the language $L(\boldsymbol{r})$ is regular. $\square$

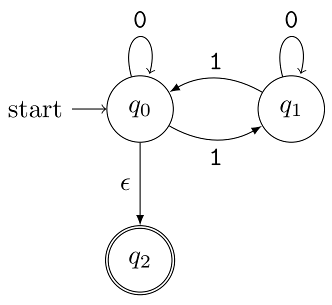
As an illustration of the state elimination algorithm we used in one direction of our proof, let us consider a small example of converting a deterministic finite automaton to a regular expression.

Example 31. Consider the following deterministic finite automaton $\mathcal{M}$:

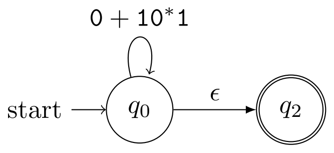


This finite automaton recognizes the language $L(\mathcal{M}) = \{w \mid w \text{ contains an even number of 1s}\}$.

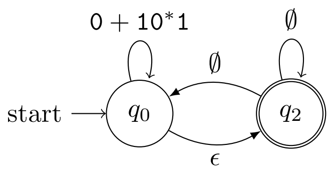
Since the initial state of $\mathcal{M}$ is also an accepting state, we begin by creating a new accepting state, converting the initial state to be nonaccepting, and then adding an epsilon transition from the initial state to our new accepting state.



We now use our state elimination algorithm to remove q_1, which is the only state that is neither initial nor accepting. There exists a single transition from q_0 to q_1 and a single transition from q_1 to q_0. Let $S_0 = \texttt{1}$, $T_0 = \texttt{1}$, $X_{00} = \texttt{0}$, and $U = \texttt{0}$. We can then eliminate the state q_1 by relabelling the loop on q_0 to use the regular expression $X_{00} + S_0U^*T_0 = \texttt{0} + \texttt{10}^*\texttt{1}$.



We add the missing transitions to obtain a finite automaton of the form specified in the proof of Theorem 30:



Consequently, the regular expression corresponding to this finite automaton is $(\texttt{0}+\texttt{10}^*\texttt{1})^*\epsilon(\emptyset+\emptyset(\texttt{0}+\texttt{10}^*\texttt{1})^*\epsilon)^*$, which simplifies to $(\texttt{0}+\texttt{10}^*\texttt{1})^*$.

Putting together all of our equivalencies, we get the following important theorem.

Theorem 32 (Kleene's theorem)**.** *A language R is regular if it satisfies any of the following equivalent properties:*

1. *There exists a deterministic finite automaton $\mathcal{M}_D$ with $L(\mathcal{M}_D) = R$;*
2. *There exists a nondeterministic finite automaton $\mathcal{M}_N$ with $L(\mathcal{M}_N) = R$; or*
3. *There exists a regular expression $\boldsymbol{r}$ with $L(\boldsymbol{r}) = R$.*

4 Proving a Language is Nonregular

By now, it should be evident that finite automata and regular expressions are nice models to use when discussing computation in the abstract. They're easy to define, easy to reason about, and they have a lot of nice properties that we can use in proofs. However, they are not the be-all and end-all of theoretical computer science. (Otherwise, this would be a rather short course!)

Both finite automata and regular expressions suffer the drawback of not having any way to store or recall data. Finite automata don't have any storage mechanism, and regular expressions don't allow for lookback. As we said in the section introducing finite automata, once the finite automaton reads a symbol and transitions to a state, it can never return to that symbol. For all intents and purposes, the symbol is lost forever, and the finite automaton doesn't even remember having read it. Likewise, once a regular expression matches a symbol in a word and moves on to the next symbol, it has no way of remembering any previous symbols that were matched.

Naturally, this means that there exist some languages that cannot be recognized by a finite automaton (or, equivalently, represented by a regular expression), and therefore such languages cannot be regular. For instance, this is the canonical example of a language that no finite automaton can recognize:

$$L_{\mathrm{a=b}} = \{\texttt{a}^n\texttt{b}^n \mid n \geq 0\}.$$

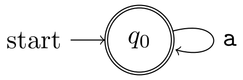
In this language, every word has an equal number of `a`s and `b`s, and all occurrences of `a` appear before the first occurrence of `b`. Some examples of words in this language are `ab`, `aaabbb`, `aaaaaabbbbbb`, and ϵ.

Why can't any finite automaton recognize this language? Because of that word *finite.* A finite automaton consists of a finite number of states, but in order to recognize this language, we would need to add a "chain" consisting of $2n$ states to accept the word $\texttt{a}^n\texttt{b}^n$ for every $n \geq 0$. Since n has no upper bound, we would need an infinite number of such "chains", and therefore an infinite number of states! No finite automaton can recognize this language, because no finite automaton has a way of keeping track of the value n or counting the symbols using only a finite number of states.

However, we can't totally rely on the claim that a finite automaton is incapable of recognizing a language if it has to count symbols. For instance, consider the language

$$L_{\mathrm{a}} = \{\texttt{a}^n \mid n \geq 0\}.$$

This language contains an infinite number of words: one word for each $n \geq 0$, exactly like in $L_{\mathrm{a=b}}$. But it's easy for a finite automaton to recognize L_{a}, and using only one state!



Thus, it should hopefully be clear that we need to take a slightly more intricate approach in order to prove a language is not regular. There are many more nonregular languages than there are regular languages, so instead of focusing on some sort of property that a nonregular language might have, let's instead find a property every regular language must have. We can then prove a language is nonregular by showing that the language *doesn't* have that property.

4.1 The Pumping Lemma for Regular Languages

The property of regular languages that we will make use of is the following: for every regular language, if we take a word in the language of sufficient length, then we can repeat (or *pump*) a middle portion of that word an arbitrary number of times and produce a new word that belongs to the same regular language. This fact, known as the *pumping lemma* for regular languages, allows us to prove a language is nonregular by contradiction; that is, by assuming the language is regular and pumping some sufficently-long word to produce a word that does not belong to the language.

The formal statement of the pumping lemma is as follows.

Lemma 33 (Pumping lemma for regular languages)**.** *For all regular languages L, there exists $p \geq 1$ where, for all $w \in L$ with $|w| \geq p$, there exists a decomposition of w into three parts $w = xyz$ such that*

1. $|y| > 0$;
2. $|xy| \leq p$; *and*
3. *for all* $i \geq 0$, $xy^iz \in L$.

Clearly, the pumping lemma contains a lot of notation and terminology to take in at once—not to mention four alternating quantifiers in a row! Let's break it down piece-by-piece to see what the lemma tells us.

- *For all regular languages L,*
  We can take any regular language L, and it will satisfy the pumping lemma.
- *there exists $p \geq 1$*
  Depending on the language L we consider, there exists a constant p for that language. We call p the *pumping constant.* (If you're curious, p is the number of states in the finite automaton recognizing L.)
- *where, for all $w \in L$ with $|w| \geq p$,*
  We can take any word from L with length at least p, and it will satisfy the pumping lemma.
- *there exists a decomposition of w into three parts $w = xyz$*
  Depending on the word w we choose, we are able to decompose w into three parts: x, y, and z. The y part is what we will use to do the pumping; the x and z parts are just the start and end parts of w that don't get pumped.
- *such that 1. $|y| > 0$;*
  This condition ensures that the y part of w is nonempty, so that we have something to pump.
- *2. $|xy| \leq p$;*
  This condition ensures that there exists some state in the finite automaton recognizing L that is visited more than once, and furthermore, we will visit that state during the computation before we finish reading the part y. (This condition is essentially an application of the pigeonhole principle.)
- *and 3. for all $i \geq 0$, $xy^iz \in L$.*
  This is the actual pumping part of the pumping lemma. This condition ensures that, no matter how

many copies of the y part we include in our word (even zero copies), the resulting word will still belong to the language.

Now that we have a greater understanding of what the pumping lemma says, let's take a look at the proof of the lemma.

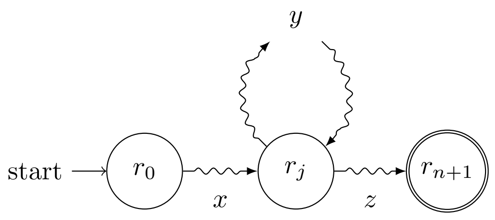
Proof of Lemma 33. Let $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite automaton recognizing the language L, and let p denote the number of states of $\mathcal{M}$.

Take a word $w = w_1 w_2 \dots w_n$ of length n from L, where $n \geq p$, and let $r_1, \dots, r_{n+1}$ be the accepting computation of $\mathcal{M}$ on w. Specifically, let $r_{i+1} = \delta(r_i, w_i)$ for all $1 \leq i \leq n$. Clearly, this accepting computation has length $n + 1 \geq p + 1$.

By the pigeonhole principle, there must exist at least two states in the first $p + 1$ states of the accepting computation that are the same. Say that the first occurrence of the same state is r_j and the second occurrence is r_ℓ. Since r_ℓ occurs within the first $p + 1$ states of the accepting computation, we know that $\ell \leq p + 1$.

Decompose the word w into parts $x = w_1 \dots w_{j-1}$, $y = w_j \dots w_{\ell-1}$, and $z = w_\ell \dots w_n$. As the part x is read, $\mathcal{M}$ transitions from state r_1 to state r_j. Likewise, as y is read, $\mathcal{M}$ transitions from r_j to r_j, and as z is read, $\mathcal{M}$ transitions from r_j to r_{n+1}. Since we are considering an accepting computation, r_{n+1} is a final state, and so $\mathcal{M}$ must accept the word xy^iz for all $i \geq 0$. Moreover, we know that $j \neq \ell$, so $|y| > 0$. Lastly, since $\ell \leq p + 1$, we have that $|xy| \leq p$. Therefore, all three conditions of the pumping lemma are satisfied. □

Diagrammatically, this proof can be reasoned about in the following way. All of the states of the finite automaton between r_0 and r_j are used to read the part x, all of the states between r_j and r_{n+1} are used to read the part z, and there exists a loop of states that both starts and ends with r_j that is used to read the part y. We can take this loop as many times as we want while reading the input word, and taking one journey around the loop corresponds to "pumping" the word once.



Alternatively, we can think of the pumping lemma as an adversarial game, where we're trying to show that some language L is nonregular while our opponent is trying to show that L is, in fact, regular. If we win the game, then L is nonregular, while if our opponent wins, then L is regular. The rules of this game are as follows, so that you can play it at the next party you attend:

Rules of the Pumping Lemma Game

1. Your opponent chooses $p \geq 1$, and they claim it is the pumping constant for L.
2. You choose a word $w \in L$ with $|w| \geq p$, and you claim this word can't be decomposed into parts $w = xyz$ that satisfy the three conditions of the pumping lemma.
3. Your opponent chooses a decomposition $w = xyz$ such that $|y| \geq 0$ and $|xy| \leq p$, satisfying the first two conditions automatically, and they claim that this decomposition will also satisfy the third condition.
4. You choose $i \geq 0$ such that $xy^iz \notin L$.

If you complete Step 4, then you win the game! Otherwise, if you can't find such an i, then you lose. Finally, if any of the claims in Steps 1–3 are false, then the person who made the claim loses.

Even though the pumping lemma looks complex, if we reduce it to a series of steps as we did here, then any proof showing that a language is nonregular simply has to follow each of the steps. As a result, nonregularity proofs tend to all have a similar structure. Let's take a look at an example of a pumping lemma proof using our canonical nonregular language, $L_{\text{a=b}}$.

Example 34. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, and consider the language $L_{\text{a=b}} = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$. We will use the pumping lemma to show that this language is nonregular.

Assume by way of contradiction that the language is regular, and let p denote the pumping constant given by the pumping lemma. We choose the word $w = \mathtt{a}^p\mathtt{b}^p$. Clearly, $w \in L_{\text{a=b}}$ and $|w| \geq p$. Thus, there exists a decomposition $w = xyz$ satisfying the three conditions of the pumping lemma.

We consider three cases, depending on the contents of the part y of the word w:

1. The part y contains only as. In this case, pumping y once to obtain the word xy^2z results in the word containing more as than bs, and so $xy^2z \notin L_{\text{a=b}}$. This violates the third condition of the pumping lemma.
2. The part y contains only bs. In this case, since the first p symbols of w are as, we must have that $|xy| > p$. This violates the second condition of the pumping lemma.
3. The part y contains both as and bs. Again, in this case, since the first p symbols of w are as, we must have that $|xy| > p$. This violates the second condition of the pumping lemma.

In all cases, one of the conditions of the pumping lemma is violated. As a consequence, the language cannot be regular.

A language doesn't necessarily have to count symbols in order to be nonregular. Since finite automata don't have any form of storage, they can't remember symbols they read earlier in an input word. This means that finite automata can't recall parts of a word, and so they can't recognize languages like $L_{\text{double}} = \{ww \mid w \in \Sigma^*\}$. Here, we prove that a similar language is nonregular: the language of palindromes, ww^{R}. (The notation w^{R} denotes the *reversal* of the word w.) Palindromes are words that read the same backward as they do forward.

Example 35. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, and consider the language $L_{\text{pal}} = \{ww^{\text{R}} \mid w \in \Sigma^*\}$. We will use the pumping lemma to show that this language is nonregular.

Assume by way of contradiction that the language is regular, and let p denote the pumping constant given by the pumping lemma. We choose the word $w = \mathtt{a}^p\mathtt{bba}^p$. Clearly, $w \in L_{\text{pal}}$ and $|w| \geq p$. Thus, there exists a decomposition $w = xyz$ satisfying the three conditions of the pumping lemma.

Since the second condition of the pumping lemma tells us that $|xy| \leq p$, it must be the case that, in any decomposition, we have $xy = \mathtt{a}^k$ for some $k \leq p$. Consequently, we have $y = \mathtt{a}^\ell$ for some $1 \leq \ell \leq k$.

If we pump y once to obtain the word xy^2z, then we obtain the word $\mathtt{a}^{p+\ell}\mathtt{bba}^p$, which is no longer a palindrome. This violates the third condition of the pumping lemma. As a consequence, the language cannot be regular.

Lastly, recall the third condition of the pumping lemma: for all $i \geq 0$, $xy^iz \in L$. The third condition allows us not only to pump *up* by adding copies of y to the word, but also to pump *down* by removing y from the word. In some cases, pumping down can help us to prove a language is nonregular.

Example 36. Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, and consider the language $L_{\text{a>b}} = \{\mathtt{a}^i\mathtt{b}^j \mid i > j\}$. We will use the pumping lemma to show that this language is nonregular.

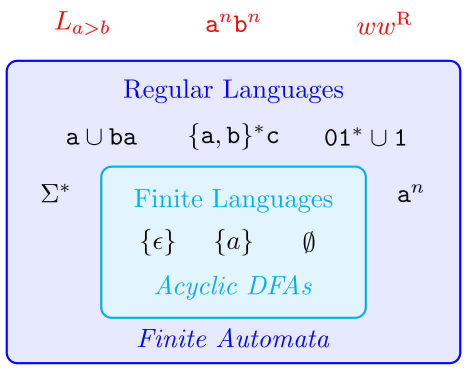
Assume by way of contradiction that the language is regular, and let p denote the pumping constant given by the pumping lemma. We choose the word $w = \mathtt{a}^{p+1}\mathtt{b}^p$. Clearly, $w \in L_{\text{a>b}}$ and $|w| \geq p$. Thus, there exists a decomposition $w = xyz$ satisfying the three conditions of the pumping lemma.

Since the second condition of the pumping lemma tells us that $|xy| \leq p$, it must be the case that, in any decomposition, we have $xy = \mathtt{a}^k$ for some $k \leq p$. Consequently, we have $y = \mathtt{a}^\ell$ for some $1 \leq \ell \leq k$.

If we pump y one or more times, then we will always end up with a word that contains more as than bs, and this word will always belong to the language $L_{\mathrm{a}>\mathrm{b}}$.

However, if we pump y down to obtain the word $xy^0z = xz$, then our word will be of the form $\mathtt{a}^{p+1-\ell}\mathtt{b}^p$. Since $\ell \geq 1$, our resultant word has at most as many as as bs, and so it no longer belongs to the language $L_{\mathrm{a}>\mathrm{b}}$. This violates the third condition of the pumping lemma. As a consequence, the language cannot be regular.

Now that we've established that there exist both regular languages and nonregular languages, we can draw a diagram to represent the theory world as we know it so far. For the time being, we're only familiar with two language classes: the class of regular languages and the class of finite languages, which is a subclass of the regular languages that we mentioned very briefly. We also only know about one machine model: finite automata.[3] As a result, our diagram admittedly isn't very interesting right now, but as we continue in the course, we will expand and add to it.



[3]Finite languages, being a subclass of the regular languages, are recognized by a special kind of deterministic finite automaton with no cycles.