St. Francis Xavier University
Department of Computer Science

CSCI 550: Approximation Algorithms
Assignment 1
Due October 6, 2022 at 1:15pm

Assignment Regulations.

- This assignment may be completed individually or in a group of two people. If you are collaborating on an assignment as a group, your group must submit exactly one joint set of answers.
- Please include your full name and email address on your submission. For groups, every member must include their full name and email address on the joint submission.
- You may either handwrite or typeset your submission. If your submission is handwritten, please ensure that the handwriting is neat and legible.

[10 marks] 1. Recall the weighted set cover problem from the lecture notes. Consider the *partial weighted set cover problem*, where we want to cover at least $p\%$ of elements in our ground set E:

> <u>Partial-Weighted-Set-Cover</u>
> Given: a ground set of elements $E = \{e_1, \dots, e_n\}$, subsets $S_1, \dots, S_m \subseteq E$, and a nonnegative weight $w_j \geq 0$ for each subset S_j
> Determine: the set $I \subseteq \{1, \dots, m\}$ that minimizes $\sum_{i \in I} w_i$ such that, for some constant $0 < p < 1$,
>
> $$\left| \bigcup_{i \in I} S_i \right| \geq p|E|$$

Describe an approximation algorithm to find a solution to the partial weighted set cover problem that has a performance guarantee of $(2 + \ln(1/(1-p))) \cdot \mathrm{OPT}$, where OPT is the value of the optimal solution to the weighted set cover problem (*not* the partial weighted set cover problem).

Hint. Adapt the greedy algorithm we have for the weighted set cover problem by modifying the stopping condition appropriately. You may find the analysis in Section 1.6 of the course textbook to be useful. You may also find the following fact useful: $\ln(x) \leq H_x \leq \ln(x) + 1$ for all integers x.

[10 marks] 2. (a) Consider the following generic linear program, denoted LP:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for all } i \in \{1, \dots, m\} \\ & x_j \geq 0, \quad \text{for all } j \in \{1, \dots, n\} \end{aligned} \tag{LP}$$

In this linear program, for each i and j, x_j is a decision variable and a_{ij}, b_i, and c_j are constants. Write the dual linear program corresponding to LP.

(b) The property of resiliency is crucial when designing and constructing a communications network. (A certain Canadian telecommunications company learned this the hard way in July 2022, when their entire network failed for a day.)

We can model network resiliency in the following way. Think of our network as an undirected graph $G = (V, E)$, where each edge $e \in E$ has an associated *cost* $c_e \geq 0$ and each pair of vertices $u, w \in V$, $u \neq w$, has a non-negative integer *connectivity requirement* r_{uw}. In the *network resiliency problem*, we must find a minimum-cost subset of edges $F \subseteq E$ such that, for all pairs of vertices $u, w \in V$, $u \neq w$, there exist at least r_{uw} edge-disjoint paths connecting u and w in our network.

Create an integer program to model the network resiliency problem, and then relax your integer program in an appropriate way to get the corresponding linear program.

Hint. One of your constraints will need to maintain the connectivity requirement by ensuring that a sufficient number of edges remain if we were to cut some subset of vertices $S \subseteq V$ from the remainder of the graph.

[10 marks] 3. Recall the set cover problem from the lecture notes. Consider the *maximum cover problem*, where we simply try to cover as many elements of the ground set E as possible:

MAXIMUM-COVER
Given: a ground set of elements $E = \{e_1, \ldots, e_n\}$, subsets $S_1, \ldots, S_m \subseteq E$, and an integer $k \geq 1$
Determine: a selection of k subsets that maximizes the number of covered elements of E

It is possible to write a rudimentary local search approximation algorithm for the maximum cover problem that starts with any solution—that is, any collection of subsets $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$—and attempts to reach a local optimum by replacing a subset from the current solution with some other subset.

Prove that the locally optimal solution A given by such an algorithm is a 2-approximation of the optimal solution O.

Hint. If the optimal solution covers $|O|$ elements and our algorithm covers $|A|$ elements, then we want to show that $|O| \leq 2 \cdot |A|$. To show this, try obtaining a bound on $|O \setminus A|$ first.

[10 marks] 4. Recall the k-center problem from the lecture notes. Consider the *k-supplier problem*, which is similar to the k-center problem but with the vertex set being partitioned into *suppliers* and *consumers*:

K-SUPPLIER
Given: an undirected graph $G = (V, E)$, a subset of suppliers $F \subseteq V$ and a subset of consumers $D = V \setminus F$, a distance $d(i, j) \geq 0$ for each pair of vertices $i, j \in V$, and an integer k
Determine: a subset of suppliers $S \subseteq F$, $\lvert S\rvert \leq k$, that minimizes the maximum distance from a supplier to a consumer

Describe an approximation algorithm to find a solution to the k-supplier problem that has a performance guarantee of $3 \cdot \mathrm{OPT}$.

Hint. Use the approximation algorithm we have for the k-center problem.