St. Francis Xavier University
Department of Computer Science

CSCI 550: Approximation Algorithms
Assignment 2
Due November 3, 2022 at 1:15pm

Assignment Regulations.

- This assignment may be completed individually or in a group of two people. If you are collaborating on an assignment as a group, your group must submit exactly one joint set of answers.
- Please include your full name and email address on your submission. For groups, every member must include their full name and email address on the joint submission.
- You may either handwrite or typeset your submission. If your submission is handwritten, please ensure that the handwriting is neat and legible.

[10 marks] 1. Recall the problem of weighted sequential scheduling on a single machine, and consider a variant of this problem where release times are not used; that is, given a set of jobs, we must schedule each job on a single machine nonpreemptively in a way that minimizes the weighted sum of completion times $\sum_{j=1}^{n} w_j C_j$.

Suppose that jobs are sorted in order of non-increasing ratio of weight to processing time: $w_1/p_1 \geq w_2/p_2 \geq \cdots \geq w_n/p_n$. Prove that our optimal strategy is to schedule job 1 first, job 2 second, and so on.

Hint. Consider what happens if two adjacent jobs are placed out of order and we swap the position of these two jobs in the ordering.

[10 marks] 2. Consider the following variant of the weighted maximum satisfiability problem:

> <u>WEIGHTED-VARIABLE-MAX-SAT</u>
> Given: n Boolean variables $x_1, \ldots, x_n$, m clauses $C_1, \ldots, C_m$ containing only positive variables, an associated weight $v_i \geq 0$ for each variable x_i, and an associated weight $w_j \geq 0$ for each clause C_j
> Determine: an assignment of true/false values for each variable x_i that maximizes the total weight of the satisfied clauses and the total weight of Boolean variables set to false

Observe that the differences between WEIGHTED-VARIABLE-MAX-SAT and WEIGHTED-MAX-SAT from the lecture notes are (i) the weights being associated to variables as well as clauses, (ii) the clauses containing only positive variables, and (iii) the additional objective of maximizing the total weight of Boolean variables set to false.

Give an integer programming formulation for this problem, and then relax it to a linear program. You may assume that you are given two decision variables: y_i, to indicate whether the variable x_i in the Boolean formula is set to true; and z_j, to indicate whether the clause C_j is satisfied.

[10 marks] 3. Consider the *data routing problem*, where we must route data packets through a ring network where all computers are connected in a circular chain. More precisely, the ring network consists of n computers numbered C_0 through C_{n-1} progressing clockwise around the ring. There exists a set D of data packets, where each data packet consists of a pair (C_i, C_j) indicating that the data is sent from computer C_i to computer C_j. The data packet can travel either clockwise or counter-clockwise around the ring network. Our objective is to route the data packets so that the maximum load on the ring network is minimized; that is, to minimize $\max_{1 \leq i \leq n} L_i$, where L_i denotes the load on the network connection between computer C_i and computer $C_{(i+1) \bmod n}$.

To obtain a solution to this problem, we can round the solution given by the following linear program:

$$\begin{array}{lll} \text{minimize} & L_i & \\ \text{subject to} & \displaystyle\sum_{P:e\in P} x_P \leq L_i & \text{for all edges } e \\ & \displaystyle\sum_{P\in P_i} x_P = 1 & \text{for all data packets } i \\ & x_P \geq 0 & \text{for all paths } P \in \mathcal{P} \end{array}$$

In this linear program, P_i represents the set of two paths (clockwise and counter-clockwise) that data packet i can follow, and $\mathcal{P} = \bigcup_i P_i$.

Describe how an approximation algorithm can round the decision variables in this linear program to obtain a solution to the problem, and show that such an algorithm gives a 2-approximation.

[10 marks] 4. The *directed weighted maximum cut problem* is closely related to the weighted maximum cut problem, but on a directed graph instead of an undirected graph.

DIRECTED-WEIGHTED-MAX-CUT
Given: a directed graph $G = (V, E)$ and an associated weight $w_{ij} \geq 0$ for each directed edge $(i, j) \in E$
Determine: a partitioning of the vertex set V into subsets U and $W = V \setminus U$ that maximizes $w = \sum_{i\in U, j\in W} w_{ij}$

In other terms, our objective is to maximize the total weight of directed edges (i, j) with $i \in U$ and $j \in W$.

Describe a $\frac{1}{4}$-approximation algorithm for the directed weighted maximum cut problem.