St. Francis Xavier University
Department of Computer Science

CSCI 356: Theory of Computing
Final Examination
December 11, 2023
2:00pm–4:30pm

Student Name: ______________________________

Email Address: ______________________________

Instructor: T. J. Smith (Section 10)

Format:

The exam is 150 minutes long. The exam consists of 7 questions worth a total of 75 marks. The exam booklet contains 10 pages, including the cover page and one blank page at the back of the exam booklet for rough work.

Reference Materials:

None.

Instructions:

1. Write your name and email address in the spaces above.
2. Answer each question either in the space provided or on a blank page. If you use a blank page to write your answer, indicate this clearly in the space provided for the question. Show all of your work.
3. Ensure that your exam booklet contains 10 pages. Do not detach any pages from your exam booklet.
4. Do not use any unauthorized reference materials or devices during this exam.
5. Sign in the space below. Your signature indicates that you understand and agree to these instructions and the university's examination policies.

Signature: ______________________________

Question	Marks	Score
1	20	
2	10	
3	8	
4	10	
5	10	
6	9	
7	8	
Total	75	

Multiple Choice

[20 marks] 1. For each of the following questions, select exactly one answer by circling the associated letter. Incorrect answers will not be penalized. Answers with more than one letter circled will be marked as incorrect.

(a) Which of the following models of computation may have a state set Q of infinite size?

A. Finite automata
B. Pushdown automata
C. Turing machines
D. None of the above

(b) Which of the following languages is **not** context-free?

A. $\{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^n \mid n, m \geq 0\}$
B. $\{\mathtt{a}^n\mathtt{b}^m \mid n \leq m \leq 2n\}$
C. $\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$
D. $\{\mathtt{a}^\ell\mathtt{b}^m\mathtt{c}^n \mid \ell = m \text{ or } m = n\}$
E. $\{\mathtt{a}^n\mathtt{b}^m \mid n, m \geq 0 \text{ and } n \neq m\}$

(c) Which of the following operations is **not** closed for the class of context-free languages?

A. Union ($\cup$)
B. Intersection ($\cap$)
C. Concatenation ($\cdot$)
D. Kleene star (*)

(d) Consider the following context-free grammar:

$$S \to \mathtt{a}X\mathtt{a} \mid \mathtt{b}X\mathtt{b} \mid \epsilon$$
$$X \to \mathtt{a} \mid \mathtt{b} \mid \epsilon$$

Which of the following words is **not** generated by this grammar?

A. `aaaaa`
B. `abaabaaba`
C. `abbaabba`
D. `abbbaa`
E. `bbabbabb`

(e) Consider a context-free grammar G in Chomsky normal form. Which of the following forms must any rule in G take?

A. $A \to BAC$ or $A \to a$ for $A, B, C \in V$ and $a \in \Sigma$.
B. $A \to BC$ or $A \to a$ for $A, B, C \in V$ and $a \in \Sigma$.
C. $A \to BAC$ or $A \to a$ for $A, B, C \in \Sigma$ and $a \in V$.
D. $A \to BC$ or $A \to a$ for $A, B, C \in \Sigma$ and $a \in V$.

(f) Which of the following is **not** equivalent in computational power to a single-tape deterministic Turing machine?

A. A nondeterministic Turing machine with sixteen tapes and sixteen input heads.

B. A deterministic Turing machine with one tape of finite length n.

C. A deterministic Turing machine with six one-way infinite tapes.

D. Microsoft PowerPoint.

(g) Let $\mathcal{M}$ be a Turing machine. What is the difference between A_{TM} and $HALT_{\mathsf{TM}}$?

A. A_{TM} asks whether $\mathcal{M}$ accepts an input word, while $HALT_{\mathsf{TM}}$ asks whether $\mathcal{M}$ does not loop forever on an input word.

B. A_{TM} applies only when $\mathcal{M}$ is nondeterministic, while $HALT_{\mathsf{TM}}$ applies only when $\mathcal{M}$ is deterministic.

C. A_{TM} can be decided using the same procedure as A_{CFG}, while $HALT_{\mathsf{TM}}$ cannot be decided at all.

D. A_{TM} is semidecidable, while $HALT_{\mathsf{TM}}$ is not semidecidable.

(h) Which of the following decision problems is decidable?

A. $\{\langle\mathcal{M}\rangle \mid \mathcal{M}$ is a Turing machine that halts on all input words$\}$.

B. $\{\langle\mathcal{M}\rangle \mid \mathcal{M}$ is a Turing machine and $L(\mathcal{M}) = \Sigma^*\}$.

C. $\{\langle\mathcal{P}, \mathcal{Q}\rangle \mid \mathcal{P}$ and $\mathcal{Q}$ are pushdown automata and $L(\mathcal{P}) = L(\mathcal{Q})\}$.

D. $\{\langle\mathcal{P}\rangle \mid \mathcal{P}$ is a pushdown automaton and $L(\mathcal{P}) = \emptyset\}$.

E. None of the above decision problems are decidable.

(i) Let L be a decidable language. Which of the following statements is **false**?

A. Both L and $\overline{L}$ are semidecidable.

B. L can always be recognized by some pushdown automaton.

C. For any language M where $M \leq_m L$, M is decidable.

D. There exists a Turing machine that halts on all instances of L.

(j) Suppose there exists a mapping reduction $X \leq_m Y$ between two decision problems X and Y. Which of the following statements is **true** for all X and Y?

A. If Y is regular, then X is regular.

B. If Y is decidable, then X is regular.

C. If Y is regular, then X is decidable.

D. If Y is undecidable, then X is undecidable.

Short Answer

[10 marks] 2. For each of the following questions, give a 1–2 sentence answer.

(a) Define a new regular expression operator "?" as follows: `a?` matches either zero occurrences or one occurrence of the symbol `a`. How can we define the "?" operator using only the three regular operations?

(b) Suppose we have a pushdown automaton with two stacks. What is the largest class of languages that this model can recognize, and why?

(c) Consider a Turing machine with an input alphabet Σ and a tape alphabet Γ. Can Σ ever be equal to Γ? Why or why not?

(d) What does it mean for a decision problem to be co-semidecidable?

(e) What is the Church–Turing thesis? (Either state the thesis itself or describe what it tells us.)

Long Answer

[8 marks] 3. Let $\Sigma = \{0, 1\}$, and consider the language

$$L = \{0^i 1^{i+j} 0^j \mid i, j \geq 0\}.$$

(a) Using the pumping lemma for regular languages, prove that L is not regular.

(b) Prove that L is context-free by creating a context-free grammar that generates all words in L.

[10 marks] 4. Construct a single-tape deterministic Turing machine $\mathcal{M}$ that decides the following language L over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$:

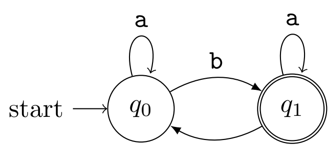
$$L = \{\mathtt{a}^i \mathtt{b}^k \mid 0 \leq i \leq k\}.$$

You should give each component of the tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$. You may optionally draw the Turing machine to illustrate your construction.

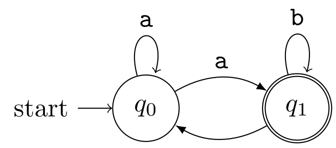
[10 marks] 5. Consider the following decision problems:

$$A_{\mathsf{NFA}} = \{\langle \mathcal{B}, w\rangle \mid \mathcal{B} \text{ is a nondeterministic finite automaton that accepts input word } w\}$$
$$EQ_{\mathsf{DFA,RE}} = \{\langle \mathcal{B}, \boldsymbol{r}\rangle \mid \mathcal{B} \text{ is a deterministic finite automaton, } \boldsymbol{r} \text{ is a regular expression, and } L(\mathcal{B}) = L(\boldsymbol{r})\}$$

Consider the following finite automata:



Finite automaton $\mathcal{B}$



Finite automaton $\mathcal{C}$

Furthermore, let $\boldsymbol{r} = \mathtt{a}^*\mathtt{b}(\mathtt{a}^*\mathtt{b})^*$ be a regular expression.

For each of the following questions, answer "yes" or "no" and justify your answer.

(a) Is $\langle \mathcal{B}, \mathtt{aaababbaa}\rangle \in A_{\mathsf{NFA}}$?

(b) Is $\langle \mathcal{C}, \mathtt{aaabb}\rangle \in A_{\mathsf{NFA}}$?

(c) Is $\langle \mathcal{B}, \boldsymbol{r}\rangle \in EQ_{\mathsf{DFA,RE}}$?

(d) Is $\langle \mathcal{C}, \boldsymbol{r}\rangle \in EQ_{\mathsf{DFA,RE}}$?

(e) Is $\langle \mathcal{B}, \mathcal{C}\rangle \in EQ_{\mathsf{DFA,RE}}$?

[9 marks] 6. (a) Let $\Sigma = \{\mathtt{0}, \mathtt{1}\}$. Consider the following decision problem:

$$ODD_{\mathsf{DFA}} = \{\langle \mathcal{B} \rangle \mid \mathcal{B} \text{ is a deterministic finite automaton that doesn't accept any word containing an odd number of } \mathtt{1}\text{s}\}.$$

Prove that ODD_{DFA} is decidable by giving a decision algorithm.

Hint. You may find the following fact useful: the class DFA is closed under intersection.

(b) Prove that $\leq_m$ is a transitive relation; that is, for any three decision problems A, B, and C, if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$.

[8 marks] 7. Consider the following decision problem:

$$SUB_{\mathsf{TM}} = \{\langle \mathcal{M}, \mathcal{N} \rangle \mid \mathcal{M} \text{ and } \mathcal{N} \text{ are Turing machines and } L(\mathcal{M}) \subseteq L(\mathcal{N})\}.$$

Prove that SUB_{TM} is undecidable by reducing from E_{TM} to SUB_{TM}.

Hint. If $L(\mathcal{M})$ is empty, what language $L(\mathcal{N})$ must we take to satisfy the subset relation?

[2 marks] *Bonus.* What was your favourite part of this course, and why?

This blank page may be used for rough work.