CSCI 355: Algorithm Design and Analysis
5. Divide and Conquer I

- *mergesort*
- *randomized quicksort*
- *median and selection*
- *closest pair of points*

Divide-and-conquer paradigm

Divide-and-conquer.
- Divide up problem into several subproblems (of the same kind).
- Solve (conquer) each subproblem recursively.
- Combine solutions to subproblems into overall solution.

Most common usage.
- Divide a problem of size n into two subproblems of size $n/2$. ← $O(n)$ time
- Solve (conquer) two subproblems recursively.
- Combine two solutions into overall solution. ← $O(n)$ time

Consequence.
- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $O(n \log n)$.

Divide-and-conquer paradigm

Divide-and-conquer.
- Divide up problem into several subproblems (of the same kind).
- Solve (conquer) each subproblem recursively.
- Combine solutions to subproblems into overall solution.

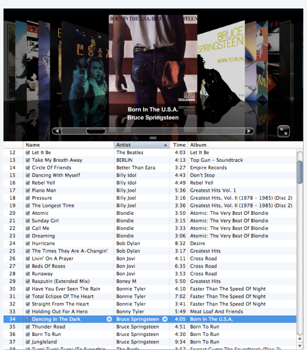
Examples.
- Mergesort and quicksort (sorting).
- Mergehull and quickhull (convex hull).
- Shamos–Hoey algorithm (closest pair).
- Median-of-medians algorithm (selection).
- Strassen's algorithm (matrix multiplication).
- Karatsuba's algorithm (integer multiplication).
- Cooley–Tukey algorithm (fast Fourier transform).

CSCI 355: Algorithm Design and Analysis
5. Divide and Conquer I

- *mergesort*
- *randomized quicksort*
- *median and selection*
- *closest pair of points*

Sorting problem

Problem. Given a list L of n elements from a totally ordered universe, rearrange them in ascending order.



Sorting applications

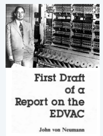
Obvious applications.
- Organize a music library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.
- Identify statistical outliers.
- Perform binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.
- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Scheduling to minimize maximum lateness.
- Minimum spanning trees (Kruskal's algorithm).
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.



input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

sort right half

A	G	L	O	R	H	I	M	S	T

merge results

A	G	H	I	L	M	O	R	S	T

Mergesort: merging

see 5A

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

sorted list A

3	7	10	a_i	18

sorted list B

2	11	b_j	20	23

merge to form sorted list C

2	3	7	10	11					

Mergesort: implementation

Input. List L of n elements from a totally ordered universe.
Output. The n elements in ascending order.

```
MERGE-SORT(L)

IF (list L has one element)
    RETURN L.

Divide the list into two halves A and B.
A ← MERGE-SORT(A).   ←—— T(n / 2)
B ← MERGE-SORT(B).   ←—— T(n / 2)
L ← MERGE(A, B).     ←—— Θ(n)
RETURN L.
```

Mergesort: proof of correctness

Proposition. Mergesort sorts any list of n elements.

Pf. [by strong induction on n]

- Base case: $n = 1$. Clearly, a list of 1 element is already sorted.
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- By the inductive hypothesis, mergesort sorts both left and right halves.
- Merging operation combines two sorted lists into a sorted whole. ▪

A useful recurrence relation

Def. $T(n)$ = max number of comparisons to mergesort a list of length n.

Recurrence.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n & \text{if } n > 1 \end{cases}$$

between $\lfloor n/2 \rfloor$ and $n - 1$ comparisons

Solution. $T(n)$ is $O(n \log_2 n)$.

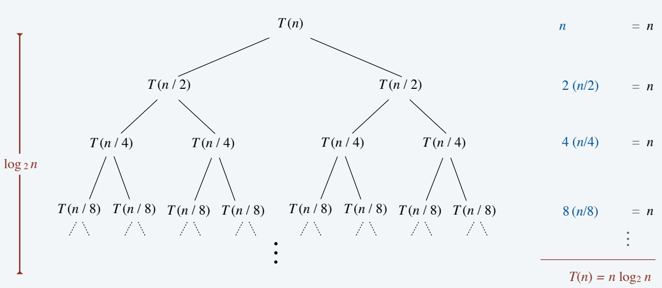
Assorted proofs. We describe several ways to solve this recurrence. Initially, we assume n is a power of 2 and replace $\le$ with $=$ in the recurrence.

Divide-and-conquer recurrence: recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{if } n > 1 \end{cases}$$



Divide-and-conquer recurrence: proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

$$T(n) \;=\; \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) \;+\; n & \text{if } n > 1 \end{cases}$$

assuming n is a power of 2

Pf. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0 = n \log_2 n$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

recurrence

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ \text{inductive hypothesis} \longrightarrow \quad &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad \blacksquare \end{aligned}$$

Mergesort: analysis

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) \le n \lceil \log_2 n \rceil$.

$$T(n) \;\le\; \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) \;+\; T(\lceil n/2 \rceil) \;+\; n & \text{if } n > 1 \end{cases}$$

no longer assuming n is a power of 2

Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n / 2 \rfloor$ and $n_2 = \lceil n / 2 \rceil$ and note that $n = n_1 + n_2$.
- Induction step: assume true for $1, 2, \ldots, n - 1$.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ \text{inductive hypothesis} \longrightarrow \quad &\le n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\le n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \\ &\le n (\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil. \quad \blacksquare \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \end{aligned}$$

$$\log_2 n_2 \le \lceil \log_2 n \rceil - 1$$

an integer

Sorting lower bound

Challenge. How to prove a lower bound for all conceivable algorithms?

Model of computation. Comparison trees.

- Can access the elements only through pairwise comparisons.
- All other operations (control, data movement, etc.) are free.

Cost model. Number of comparisons.

Q. Realistic model?

A1. Yes: Java, Python, C++, …

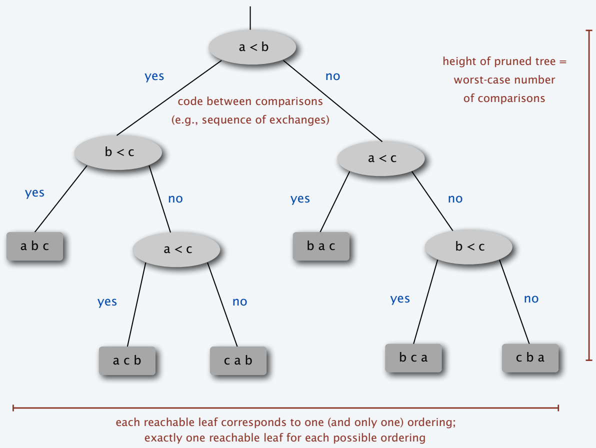
A2. Yes: Mergesort, insertion sort, quicksort, heapsort, …

A3. No: Bucket sort, radix sorts, …

sort(**, key=None, reverse=False*)

This method sorts the list in place, using only < comparisons between items. Exceptions are not suppressed – if any comparison operations fail, the entire sort operation will fail (and the list will likely be left in a partially modified state).

Comparison tree (for 3 distinct keys a, b, and c)



Sorting lower bound

Theorem. Any deterministic comparison-based sorting algorithm must make $\Omega(n \log n)$ comparisons in the worst-case.

Pf. [information theoretic]

- Assume array consists of n distinct values a_1 through a_n.
- Worst-case number of comparisons = height h of pruned comparison tree.
- Binary tree of height h has $\leq 2^h$ leaves.
- n! different orderings $\Rightarrow$ n! reachable leaves.

$$
\begin{aligned}
2^h &\geq \text{\# reachable leaves} = n\,! \\
\Rightarrow\ h &\geq \log_2(n!) \\
&\geq n \log_2 n - n / \ln 2 \quad \blacksquare
\end{aligned}
$$

Stirling's formula

Note. Lower bound can be extended to include randomized algorithms.

CSCI 355: Algorithm Design and Analysis
5. Divide and Conquer I

- mergesort
- *randomized quicksort*
- median and selection
- closest pair of points

Randomized quicksort

- Pick a random pivot element $p \in A$.
- 3-way partition the array into L, M, and R.
- Recursively sort both L and R.

the array A	7	6	12	3	11	8	9	1	4	10	2
		p									
partition A	3	1	4	2	6	7	12	11	8	9	10
sort L	1	2	3	4	6	7	12	11	8	9	10
sort R	1	2	3	4	6	7	8	9	10	11	12
the sorted array A	1	2	3	4	6	7	8	9	10	11	12

Randomized quicksort: proof of correctness

Proposition. Quicksort sorts any array of n elements.

Pf. [by strong induction on n]

- Base case: $n = 1$. Clearly, a list of 1 element is already sorted.
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- 3-way partitioning rearranges array around pivot p so that
  - elements smaller than p in left subarray L
  - elements equal to p in middle subarray M
  - elements larger than p in right subarray R
- All elements in L must appear before all elements in M, which must appear before all elements in R.
- By inductive hypothesis, quicksort sorts both L and R. ▪

M is nonempty

3	1	4	2	6	7	12	11	8	9	10
L				M	R					

Randomized quicksort: implementation

see 5B

- Pick a random pivot element $p \in A$.
- 3-way partition the array into L, M, and R.
- Recursively sort both L and R.

```
RANDOMIZED-QUICKSORT(A)

IF (array A has zero or one element)
   RETURN.
Pick pivot p ∈ A uniformly at random.
(L, M, R) ← PARTITION-3-WAY(A, p).      ← Θ(n)
RANDOMIZED-QUICKSORT(L).                ← T(i)
RANDOMIZED-QUICKSORT(R).                ← T(n – i – 1)
```

new analysis required
(i is a random variable—depends on p)

Randomized quicksort: analysis

Proposition. The expected number of comparisons to quicksort an array of n distinct elements $a_1 < a_2 < \cdots < a_n$ is $O(n \log n)$.

Pf. Consider BST representation of pivot elements.

- a_i and a_j are compared once iff one is an ancestor of the other.
- **Pr** [a_i and a_j are compared] $= 2 \,/\, (j - i + 1)$, where $i < j$.
- Expected number of comparisons $= \underbrace{\sum_{i=1}^{n} \sum_{j=i+1}^{n}}_{\text{all pairs } i \text{ and } j} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n} \sum_{j=2}^{n-i+1} \frac{1}{j}$

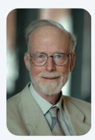
$$\leq 2n \sum_{j=1}^{n} \frac{1}{j}$$

$$\underbrace{\leq}_{\text{harmonic sum}} 2n\,(\ln n + 1) \quad ▪$$

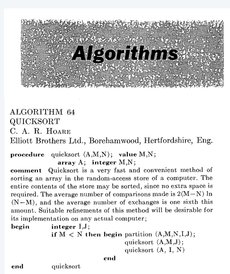
Remark. # of comparisons decreases only if there are equal elements.

Randomized quicksort

- Tony Hoare invented quicksort to translate Russian into English.
  [after realizing insertion sort wasn't good enough!]
- Learned Algol 60 (and recursion) to implement it.
- Published the algorithm in CACM.



Tony Hoare



Algorithms

ALGORITHM 64
QUICKSORT
C. A. R. Hoare
Elliott Brothers Ltd., Borehamwood, Hertfordshire, Eng.

```
procedure quicksort (A,M,N); value M,N;
          array A; integer M,N;
comment Quicksort is a very fast and convenient method of
sorting an array in the random-access store of a computer. The
entire contents of the store may be sorted, since no extra space is
required. The average number of comparisons made is 2(M—N) ln
(N—M), and the average number of exchanges is one sixth this
amount. Suitable refinements of this method will be desirable for
its implementation on any actual computer;
begin     integer I,J;
          if M < N then begin partition (A,M,N,I,J);
                              quicksort (A,M,J);
                              quicksort (A, I, N)
                         end
end       quicksort
```

Communications of the ACM (July 1961)

CSCI 355: Algorithm Design and Analysis
5. Divide and Conquer I

- mergesort
- randomized quicksort
- *median and selection*
- closest pair of points

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n+1) / 2 \rfloor$.
- $O(n)$ comparisons for min or max.
- $O(n \log n)$ comparisons by sorting.
- $O(n \log k)$ comparisons with a binary heap. ← max heap with k smallest

Applications. Order statistics; finding the "top k"; bottleneck paths, …

Q. Can we do it with $O(n)$ comparisons?
A. Yes! Selection is easier than sorting.

Randomized quickselect

see 5B

- Pick a random pivot element $p \in A$.
- 3-way partition the array into L, M, and R.
- Recur in one subarray—the one containing the k^{th} smallest element.

QUICK-SELECT(A, k)

Pick pivot $p \in A$ uniformly at random.
$(L, M, R) \leftarrow$ PARTITION-3-WAY(A, p). ← $\Theta(n)$
IF $(k \le |L|)$ RETURN QUICK-SELECT(L, k). ← $T(i)$
ELSE IF $(k > |L| + |M|)$ RETURN QUICK-SELECT$(R, k - |L| - |M|)$ ← $T(n - i - 1)$
ELSE IF $(k = |L|)$ RETURN p.

Randomized quickselect: analysis

Intuition. Split candy bar uniformly ⇒ expected size of larger piece is 3/4.

$$T(n) \le T(3n/4) + n \quad \Rightarrow \quad T(n) \le 4n$$

not rigorous: can't assume $E[T(i)] \le T(E[i])$

Def. $T(n, k)$ = expected # comparisons to select k^{th} smallest in array of length $\le n$.
Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \le 4n$.
Pf. [by strong induction on n]

- Assume true for $1, 2, \ldots, n - 1$.
- $T(n)$ satisfies the following recurrence:

can assume we always recur of larger of two subarrays since $T(n)$ is monotone non-decreasing

$$\begin{aligned} T(n) &\le n + 1/n\, [\; 2T(n/2) + \ldots + 2T(n-3) + 2T(n-2) + 2T(n-1) \;] \\ &\le n + 1/n\, [\; 8(n/2) + \ldots + 8(n-3) + 8(n-2) + 8(n-1) \;] \\ &\le n + 1/n\, (3n^2) \\ &= 4n. \quad \blacksquare \end{aligned}$$

inductive hypothesis

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst-case linear time

Goal. Find a pivot element p that divides the list of n elements into two pieces so that each piece is guaranteed to have $\leq$ 7/10 n elements.

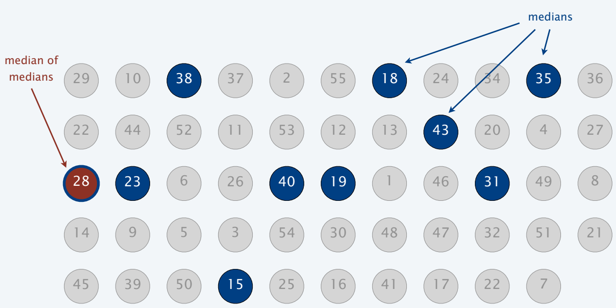
Q. How to find approximate median in linear time?
A. Recursively compute median of sample of $\leq$ 2/10 n elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ \underbrace{T(7/10\, n) + T(2/10\, n)}_{\text{two subproblems of different sizes!}} + \Theta(n) & \text{otherwise} \end{cases}$$

$$\Rightarrow \underbrace{T(n) \in \Theta(n)}_{\text{we'll need to show this}}$$

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n / 5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.



n = 54

Median-of-medians selection algorithm

MOM-SELECT(A, k)

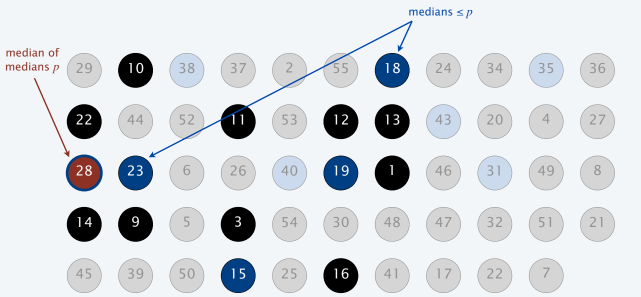
$n \leftarrow |A|$.
IF ($n < 50$)
  RETURN k^{th} smallest of element of A via mergesort.

Group A into $\lfloor n / 5 \rfloor$ groups of 5 elements each (ignore leftovers).
$B \leftarrow$ median of each group of 5.
$p \leftarrow$ MOM-SELECT(B, $\lfloor n / 10 \rfloor$) ← median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY(A, p).
IF ($k \leq |L|$) RETURN MOM-SELECT(L, k).
ELSE IF ($k > |L| + |M|$) RETURN MOM-SELECT(R, $k - |L| - |M|$)
ELSE RETURN p.

Median-of-medians selection algorithm: analysis

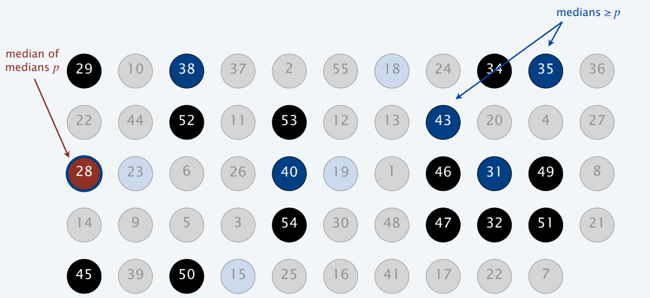
- At least half of 5-element medians $\leq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.



n = 54

Median-of-medians selection algorithm: analysis

- At least half of 5-element medians $\geq p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\geq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.



n = 54

Median-of-medians selection algorithm: recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # of comparisons on any array of n elements.

$$C(n) \;\leq\; \underbrace{C(\lfloor n/5 \rfloor)}_{\text{median of medians}} \;+\; \underbrace{C(n - 3\lfloor n/10 \rfloor)}_{\text{recursive select}} \;+\; \tfrac{11}{5}n$$

computing median of 5 (≤ 6 compares per group)

partitioning ($\leq n$ compares)

Intuition.

- $C(n)$ is going to be at least linear in $n \Rightarrow C(n)$ is super-additive.
- Ignoring floors, this implies that $C(n) \;\leq\; C(n/5 + n - 3n/10) + 11/5\, n$
  $= C(9n/10) + 11/5\, n$
  $\Rightarrow C(n) \leq 22n.$

Median-of-medians selection algorithm: recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # comparisons on any array of $\le n$ elements.
- $T(n)$ is monotone non-decreasing, but $C(n)$ is not!

$$T(n) \;\le\; \begin{cases} 6n & \text{if } n < 50 \\ \max\{\, T(n-1),\; T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + \frac{11}{5}n) \,\} & \text{if } n \ge 50 \end{cases}$$

Claim. $T(n) \le 44\,n$.

Pf. [by strong induction]

- Base case: $T(n) \le 6n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- Induction step: for $n \ge 50$, we have either $T(n) \le T(n-1) \le 44\,n$ or

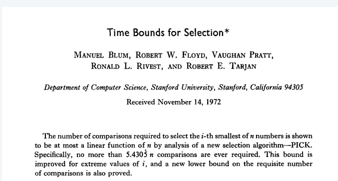
$$\begin{aligned} T(n) &\le T(\lfloor n/5 \rfloor) + T(n - 3\lfloor n/10 \rfloor) + 11/5\,n \\ &\le 44\,(\lfloor n/5 \rfloor) + 44\,(n - 3\lfloor n/10 \rfloor) + 11/5\,n \\ &\le 44\,(n/5) + 44\,n - 44\,(n/4) + 11/5\,n \\ &= 44\,n. \;\blacksquare \end{aligned}$$

inductive hypothesis

for $n \ge 50$, $3\lfloor n/10 \rfloor \ge n/4$

Linear-time selection retrospective

Proposition. [Blum–Floyd–Pratt–Rivest–Tarjan 1973] There exists a comparison-based selection algorithm whose worst-case running time is $O(n)$.


Time Bounds for Selection*

Manuel Blum, Robert W. Floyd, Vaughan Pratt, Ronald L. Rivest, and Robert E. Tarjan

Department of Computer Science, Stanford University, Stanford, California 94305

Received November 14, 1972

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm—PICK. Specifically, no more than $5.4305\,n$ comparisons are ever required. This bound is improved for extreme values of i, and a new lower bound on the requisite number of comparisons is also proved.

Theory.

- Optimized version of BFPRT: $\le 5.4305\,n$ comparisons.
- Upper bound: [Dor–Zwick 1995] $\le 2.95\,n$ comparisons.
- Lower bound: [Dor–Zwick 1999] $\ge (2 + 2^{-80})\,n$ comparisons.

Practice. BFPRT constants too large to be useful.

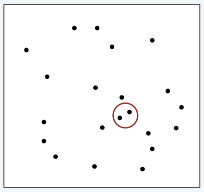
CSCI 355: Algorithm Design and Analysis
5. Divide and Conquer I

Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

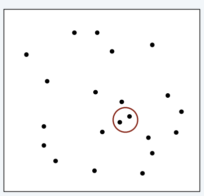
- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of $\underbrace{\text{nearest neighbor, Euclidean MST, Voronoi}}_{\text{fast closest pair inspired fast algorithms for these problems}}$.



Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

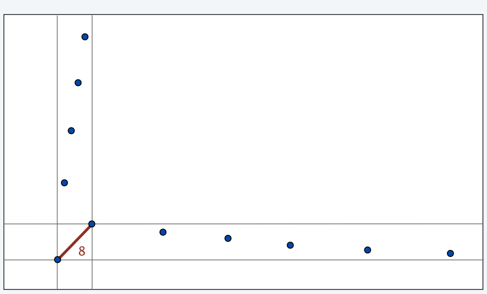
Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.
Non-degeneracy assumption. No two points have the same x-coordinate.



Closest pair of points: first attempt

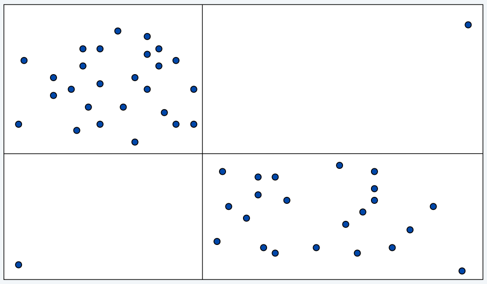
Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: second attempt

Plan. Subdivide region into 4 quadrants.
Obstacle. Impossible to ensure $n/4$ points in each piece.

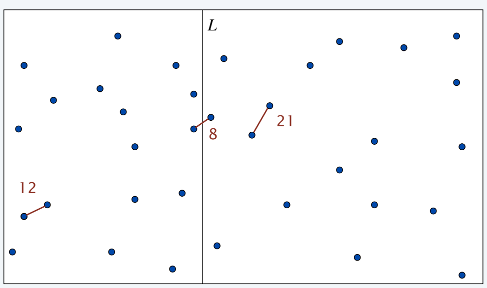


53

Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

seems like $\Theta(n^2)$

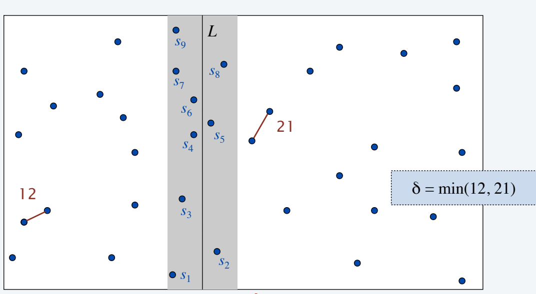


54

How to find closest pair with one point on each side?

- Assume that distance < δ.
- Observation: suffices to consider only those points within δ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Check distances of only those points within 7 positions in sorted list!

why?



56

Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR($p_1, p_2, \ldots, p_n$)

Compute vertical line L such that half the points are on each side of the line. ← $O(n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR(points in left half). ← $T(\lfloor n / 2 \rfloor)$

$\delta_2 \leftarrow$ CLOSEST-PAIR(points in right half). ← $T(\lceil n / 2 \rceil)$

$\delta \leftarrow \min \{ \delta_1 , \delta_2 \}$.

$A \leftarrow$ list of all points closer than δ to line L. ← $O(n)$

Sort points in A by y-coordinate. ← $O(n \log n)$

Scan points in A in y-order and compare distance between each point and next 7 neighbors. If any of these distances is less than δ, update δ. ← $O(n)$

RETURN δ.

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding a closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

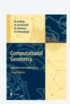
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) \ + \ T(\lceil n/2 \rceil) \ + \ O(n \log n) & \text{if } n > 1 \end{cases}$$

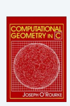
Digression: computational geometry

Ingenious divide-and-conquer algorithms for core geometric problems.

problem	brute	clever
closest pair	$O(n^2)$	$O(n \log n)$
convex hull	$O(n^2)$	$O(n \log n)$
farthest pair	$O(n^2)$	$O(n \log n)$
Delaunay/Voronoi	$O(n^4)$	$O(n \log n)$
Euclidean MST	$O(n^2)$	$O(n \log n)$

running time to solve a 2D problem with n points

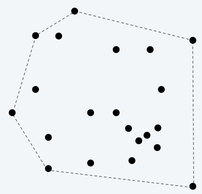




Note. 3D and higher dimensions test limits of our ingenuity.

Convex hull

The convex hull of a set of n points is the smallest perimeter fence enclosing the points.

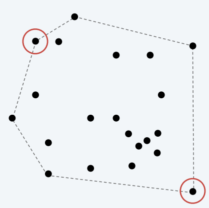


Equivalent definitions.

- Smallest area convex polygon enclosing the points.
- Intersection of all convex set containing all the points.

Farthest pair

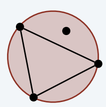
Given n points in the plane, find a pair of points with the largest Euclidean distance between them.



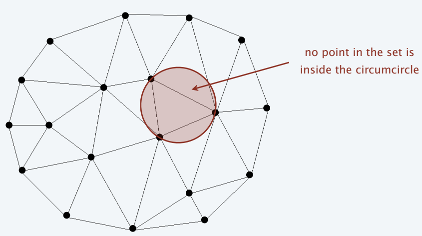
Fact. Points in farthest pair are extreme points on convex hull.

Delaunay triangulation

The Delaunay triangulation is a triangulation of n points in the plane such that no point is inside the circumcircle of any triangle.



point inside circumcircle of 3 points



Delaunay triangulation of 19 points

Some useful properties.

- No edges cross.
- Among all triangulations, it maximizes the minimum angle.
- Contains an edge between each point and its nearest neighbor.

Euclidean MST

Given n points in the plane, find MST connecting them.
[distances between point pairs are Euclidean distances]

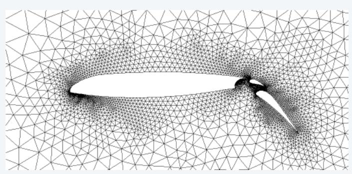
Fact. Euclidean MST is subgraph of Delaunay triangulation.
Implication. We can compute the Euclidean MST in $O(n \log n)$ time.

- Compute Delaunay triangulation.
- Compute MST of Delaunay triangulation. ← it's planar ($\leq 3n$ edges)

Computational geometry applications

Applications.

- Robotics.
- VLSI design.
- Data mining.
- Medical imaging.
- Computer vision.
- Scientific computing.
- Finite-element meshing.
- Astronomical simulation.
- Models of physical world.
- Geographic information systems.
- Computer graphics (movies, games, virtual reality).



airflow around an aircraft wing

http://www.ics.uci.edu/~eppstein/geom.html