CSCI 355: Algorithm Design and Analysis
3. Greedy Algorithms I

- *coin changing*
- *interval scheduling*
- *interval partitioning*
- *scheduling to minimize lateness*

Coin changing

Goal. Given Canadian currency denominations { 1, 5, 10, 25, 100 }, devise a method to pay amount to customer using fewest coins.

Ex. 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex. $2.89.

Cashier's algorithm

At each iteration, add coin of the largest value that does not take us past the amount to be paid.

CASHIERS-ALGORITHM $(x, c_1, c_2, \ldots, c_n)$

SORT n coin denominations so that $0 < c_1 < c_2 < \ldots < c_n$.
$S \leftarrow \emptyset$. ⟵ multiset of coins selected
WHILE $(x > 0)$
  $k \leftarrow$ largest coin denomination c_k such that $c_k \le x$.
  IF (no such k)
    RETURN *"no solution."*
  ELSE
    $x \leftarrow x - c_k$.
    $S \leftarrow S \cup \{ k \}$.
RETURN S.

Cashier's algorithm (for arbitrary coin denominations)

Q. Is cashier's algorithm optimal for any set of denominations?

A. No. Consider Canadian postage: { 1, 10, 22, 34, 71, 100, 250 }.

- Cashier's algorithm: 142¢ = 100 + 34 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
- Optimal: 142¢ = 71 + 71

Remark. It may not even lead to a feasible solution if $c_1 > 1$: 7, 8, 9.

- Cashier's algorithm: 15¢ = 9 + ?
- Optimal: 15¢ = 7 + 8

Properties of any optimal solution (for Canadian coin denominations)

Property. Number of pennies ≤ 4.

Pf. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.

Pf. Replace 2 nickels with 1 dime.

Property. Number of quarters ≤ 3.

Pf. Replace 4 quarters with 1 dollar.

Property. Number of nickels + number of dimes ≤ 2.

Pf.

- Recall: we need at most 1 nickel.
- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.

dollars (100¢) | quarters (25¢) | dimes (10¢) | nickels (5¢) | pennies (1¢)

Optimality of cashier's algorithm (for Canadian coin denominations)

Theorem. Cashier's algorithm is optimal for Canadian coins { 1, 5, 10, 25, 100 }.

Pf. [by induction on amount to be paid x]

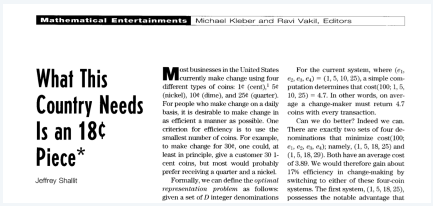
- Consider optimal way to change $c_k \le x < c_{k+1}$: greedy algorithm takes coin k.
- We claim that any optimal solution must take coin k.
  - If not, the solution needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x.
  - By the table below, no optimal solution can do this.
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by cashier's algorithm. ▪

k	c_k	all optimal solutions must satisfy	max value of coin denominations $c_1, c_2, \ldots, c_{k-1}$ in any optimal solution
1	1	$P \le 4$	–
2	5	$N \le 1$	4
3	10	$N + D \le 2$	4 + 5 = 9
4	25	$Q \le 3$	20 + 4 = 24
5	100	*no limit*	75 + 24 = 99

Room for improvement?

- Using currency denominations { 1, 5, 10, 25 }, any transaction up to $1 results in 4.7 coins being returned on average.
- If we replace the 10¢ coin with an 18¢ coin, then 3.89 coins are returned on average!
- Using currency denominations { 1, 5, 10, 25, 100, 200 }, any transaction up to $5 results in 5.9 coins being returned on average.
- If we add an 83¢ coin, then 4.578 coins are returned on average!

Ray Kuo


Mathematical Entertainments Michael Kleber and Ravi Vakil, Editors

What This Country Needs Is an 18¢ Piece*

Jeffrey Shallit

Most businesses in the United States currently make change using four different types of coins: 1¢ (cent),[1] 5¢ (nickel), 10¢ (dime), and 25¢ (quarter). For people who make change on a daily basis, it is desirable to make change in as efficient a manner as possible. One criterion for efficiency is to use the smallest number of coins. For example, to make change for 30¢, one could, at least in principle, give a customer 30 1-cent coins, but most would probably prefer receiving a quarter and a nickel.

Formally, we can define the *optimal representation problem* as follows: given a set of D integer denominations

For the current system, where $(c_1, c_2, c_3, c_4) = (1, 5, 10, 25)$, a simple computation determines that $\text{cost}(100; 1, 5, 10, 25) = 4.7$. In other words, on average a change-maker must return 4.7 coins with every transaction.

Can we do better? Indeed we can. There are exactly two sets of four denominations that minimize $\text{cost}(100; c_1, c_2, c_3, c_4)$; namely, $(1, 5, 18, 25)$ and $(1, 5, 18, 29)$. Both have an average cost of 3.89. We would therefore gain about 17% efficiency in change-making by switching to either of these four-coin systems. The first system, $(1, 5, 18, 25)$, possesses the notable advantage that

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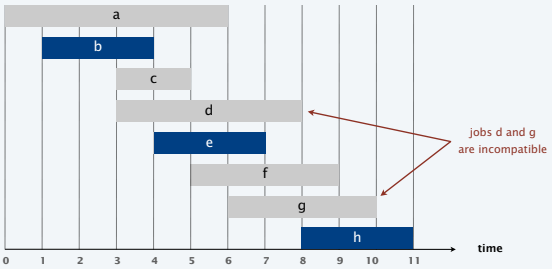
Interval scheduling

- Job j starts at time s_j and finishes at time f_j.
- Two jobs are compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval scheduling: earliest-finish-time-first algorithm

EARLIEST-FINISH-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT jobs by finish times and renumber so that $f_1 \le f_2 \le \ldots \le f_n$.

$S \leftarrow \emptyset$. ← set of jobs selected

FOR $j = 1$ TO n

  IF (job j is compatible with S)

    $S \leftarrow S \cup \{ j \}$.

RETURN S.

Proposition. We can implement earliest-finish-time first in $O(n \log n)$ time.

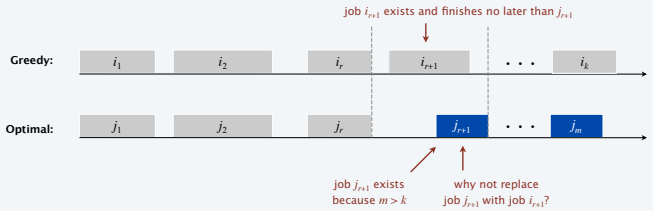
- Keep track of the job j^* that was added last to S.
- Job j is compatible with S iff $s_j \ge f_{j^*}$.
- Sorting by finish times takes $O(n \log n)$ time.

Interval scheduling: analysis of earliest-finish-time-first algorithm

Theorem. The earliest-finish-time-first algorithm is optimal.

Pf. [by contradiction]

- Assume the greedy earliest-finish-time approach is not optimal, and see what happens.
- Let $i_1, i_2, \ldots i_k$ denote the set of jobs selected by the greedy earliest-finish-time approach.
- Let $j_1, j_2, \ldots j_m$ denote the set of jobs in an optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



Interval scheduling: analysis of earliest-finish-time-first algorithm

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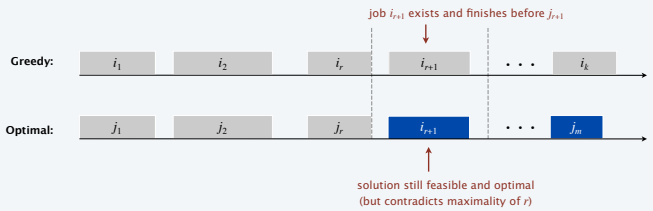

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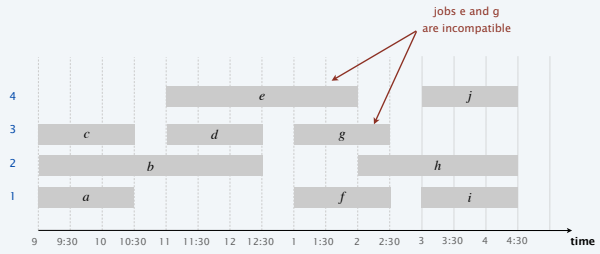
Interval partitioning

- Lecture j starts at time s_j and finishes at time f_j.
- Goal: find the minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.

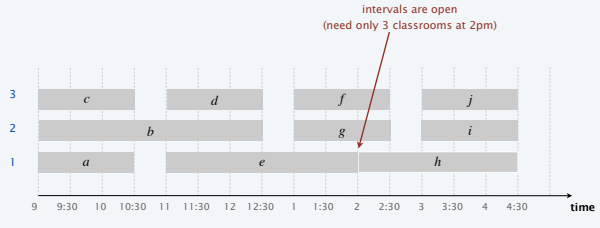
Ex. This schedule uses 4 classrooms to schedule 10 lectures.



Interval partitioning

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.

Ex. This schedule uses 3 classrooms to schedule 10 lectures.



Interval partitioning: earliest-start-time-first algorithm

EARLIEST-START-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT lectures by start times and renumber so that $s_1 \le s_2 \le \ldots \le s_n$.

$d \leftarrow 0$. ← number of allocated classrooms

FOR $j = 1$ TO n

  IF (lecture j is compatible with some classroom)

    Schedule lecture j in any such classroom k.

  ELSE

    Allocate a new classroom $d + 1$.

    Schedule lecture j in classroom $d + 1$.

    $d \leftarrow d + 1$.

RETURN schedule.

Interval partitioning: earliest-start-time-first algorithm

Proposition. We can implement earliest-start-time-first in $O(n \log n)$ time.

Pf.

- Sorting by start times takes $O(n \log n)$ time.
- Store classrooms in a priority queue (key = finish time of its last lecture).
  - To allocate a new classroom, INSERT classroom onto priority queue.
  - To schedule lecture j in classroom k, INCREASE-KEY of classroom k to f_j.
  - To determine whether lecture j is compatible with any classroom, compare s_j to FIND-MIN
- Total number of priority queue operations is $O(n)$; each takes $O(\log n)$ time. ▪

Remark. This implementation chooses a classroom k whose finish time of its last lecture is the earliest.

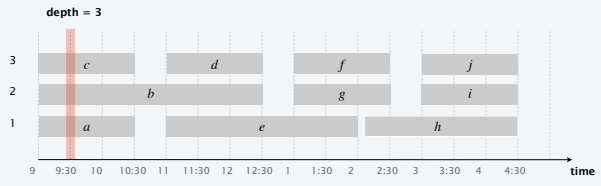
Interval partitioning: lower bound on optimal solution

Def. The depth of a set of open intervals is the maximum number of intervals that contain any given point.

Key observation. Number of classrooms needed ≥ depth.

Q. Is the minimum number of classrooms needed always equal to the depth?

A. Yes! Moreover, the earliest-start-time-first algorithm finds a schedule where the number of classrooms is equal to the depth.



Interval partitioning: analysis of earliest-start-time-first algorithm

Observation. The earliest-start-time-first algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Earliest-start-time-first algorithm is optimal.

Pf.

- Let d = number of classrooms that the earliest-start-time-first algorithm allocates.
- Classroom d was opened because we needed to schedule a lecture, say j, that was incompatible with a lecture in each of the $d - 1$ other classrooms.
- Thus, these $d - 1$ lectures all end after s_j.
- Since we sorted by start time, each of these incompatible lectures start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq d$ classrooms. ▪

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Scheduling to minimize lateness

- A single resource processes one job at a time.
- Each job j requires t_j units of processing time and is due at time d_j.
- If job j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- The lateness of a job j is $\ell_j = \max\{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize the maximum lateness $L = \max_j \ell_j$.

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

Minimizing lateness: earliest-deadline-first

EARLIEST-DEADLINE-FIRST $(n, t_1, t_2, \ldots, t_n, d_1, d_2, \ldots, d_n)$

SORT jobs by due times and renumber so that $d_1 \le d_2 \le \ldots \le d_n$.

$t \leftarrow 0$.

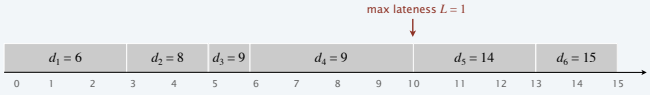
FOR $j = 1$ TO n

  Assign job j to interval $[t, t + t_j]$.

  $s_j \leftarrow t$; $f_j \leftarrow t + t_j$.

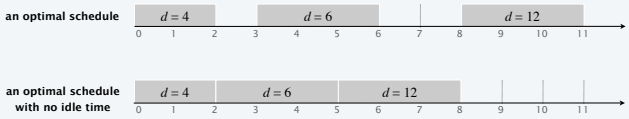
  $t \leftarrow t + t_j$.

RETURN intervals $[s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n]$.



Minimizing lateness: no idle time

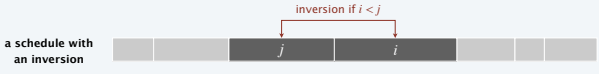
Observation 1. There exists an optimal schedule with no idle time.



Observation 2. The earliest-deadline-first schedule has no idle time.

Minimizing lateness: inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that $i < j$ but j is scheduled before i.



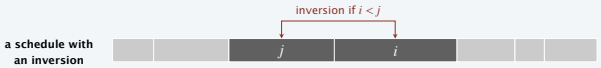
recall: we assume the jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$

Observation 3. The earliest-deadline-first schedule is the unique idle-free schedule with no inversions.

1	2	3	4	5	6	…	n

Minimizing lateness: inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: $i < j$ but j is scheduled before i.



recall: we assume the jobs are numbered so that $d_1 \le d_2 \le \ldots \le d_n$

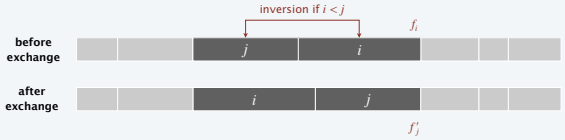
Observation 4. If an idle-free schedule has an inversion, then it has an adjacent inversion.

Pf. ← two inverted jobs scheduled consecutively

- Let i–j be a closest inversion.
- Let k be the element immediately to the right of j.
- Case 1. [$j > k$] Then j–k is an adjacent inversion.
- Case 2. [$j < k$] Then i–k is a closer inversion since $i < j < k$. ※

Minimizing lateness: inversions

Def. Given a schedule S, an inversion is a pair of jobs i and j such that: $i < j$ but j is scheduled before i.



Key claim. Exchanging two adjacent inverted jobs i and j reduces the number of inversions by 1 and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be the lateness afterwards.

- $\ell'_k = \ell_k$ for all $k \ne i, j$.
- $\ell'_i \le \ell_i$.
- If job j is late, $\ell'_j = f'_j - d_j$ ← definition
  $= f_i - d_j$ ← j now finishes at time f_i
  $\le f_i - d_i$ ← $i < j \rightarrow d_i \le d_j$
  $\le \ell_i$. ← definition

Minimizing lateness: analysis of earliest-deadline-first algorithm

Theorem. The earliest-deadline-first schedule S is optimal.

Pf. [by contradiction] ← optimal schedule can have inversions

Define S^* to be an optimal schedule with the fewest inversions.

- We can assume S^* has no idle time. ← Observation 1
- Case 1. [S^* has no inversions] Then $S = S^*$. ← Observation 3
- Case 2. [S^* has an inversion]
  - Let i–j be an adjacent inversion. ← Observation 4
  - Exchanging jobs i and j decreases the number of inversions by 1 without increasing the max lateness. ← key claim
  - This contradicts the "fewest inversions" part of the definition of S^*. ※

Greedy analysis strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm.

Structure. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Gale–Shapley, Kruskal, Prim, Dijkstra, Huffman, ...