St. Francis Xavier University
Department of Computer Science

CSCI 356: Theory of Computing
Final Examination
December 16, 2024
2:00pm–4:30pm

Student Name: ______________________________

Email Address: ______________________________

Instructor: T. J. Smith (Section 10)

Format:

The exam is 150 minutes long. The exam consists of 6 questions worth a total of 70 marks. The exam booklet contains 9 pages, including the cover page and one blank page at the back of the exam booklet for rough work.

Reference Materials:

None.

Instructions:

1. Write your name and email address in the spaces above.
2. Answer each question either in the space provided or on a blank page. If you use a blank page to write your answer, indicate this clearly in the space provided for the question. Show all of your work.
3. Ensure that your exam booklet contains 9 pages. Do not detach any pages from your exam booklet.
4. Do not use any unauthorized reference materials or devices during this exam.
5. Sign in the space below. Your signature indicates that you understand and agree to these instructions and the university's examination policies.

Signature: ______________________________

Question	Marks	Score
1	20	
2	10	
3	10	
4	10	
5	12	
6	8	
Total	70	

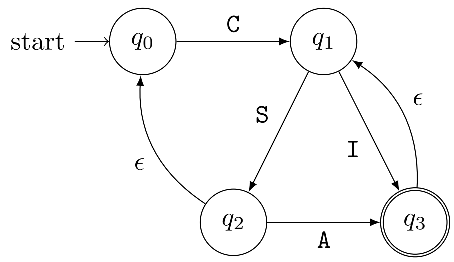
Multiple Choice

[20 marks] 1. For each of the following questions, select exactly one answer by circling the associated letter. Incorrect answers will not be penalized. Answers with more than one letter circled will be marked as incorrect.

(a) Which of the following statements about the empty word ϵ and the empty language $\emptyset$ is **false**?

A. For any language L, $L \cup \emptyset = L$.
B. For any language L, $\epsilon \cdot L = L$.
C. $\emptyset^* = \emptyset$.
D. $\epsilon^* = \{\epsilon\}$.

(b) Consider the following nondeterministic finite automaton:



Which of the following words is **not** in the language of this finite automaton?

A. `CSCI`
B. `CISCI`
C. `CSASA`
D. `CSISA`

(c) Let L be the language corresponding to the regular expression $(\texttt{1}^*\texttt{01}+\texttt{10}^*\texttt{1})^*$. Which of the following words is in L?

A. `0110`
B. `10101`
C. `11001`
D. `100100`

(d) In which order should we apply the steps to convert a context-free grammar to its equivalent Chomsky normal form?

A. **DEL**, **BIN**, **TERM**, **UNIT**, **START**
B. **START**, **TERM**, **BIN**, **DEL**, **UNIT**
C. **START**, **UNIT**, **BIN**, **TERM**, **DEL**
D. **TERM**, **DEL**, **UNIT**, **START**, **BIN**

(e) For a grammar in Chomsky normal form, which of the following rule forms is **not** admissible?

A. $A \to SB$
B. $A \to \texttt{c} \mid BC$
C. $C \to \texttt{b}$
D. $S \to CD \mid \epsilon$

(f) Which of the following statements about Turing machines is **false**?

A. The input alphabet Σ and the tape alphabet Γ are always different.

B. A Turing machine must consist of at least two states.

C. A deterministic Turing machine can simulate a nondeterministic computation.

D. Adding a second tape to a Turing machine allows it to recognize more languages than a machine with one tape.

(g) Which of the following statements is **true**?

A. If a language L is semidecidable, then its complement $\overline{L}$ is also semidecidable.

B. A language L is decidable if and only if both L and $\overline{L}$ are semidecidable.

C. A language L is decidable if and only if L is not semidecidable.

D. A language L is semidecidable if and only if L is undecidable.

(h) Which of the following statements about a semidecidable language is **false**?

A. There exists a Turing machine that always accepts words in the language.

B. The language is undecldable.

C. We can recognize the language using a universal Turing machine.

D. Any Turing machine recognizing the language halts on all input words.

(i) Which of the following decision problems is undecidable?

A. $\{\langle \mathcal{M} \rangle \mid \mathcal{M}$ is a nondeterministic finite automaton that accepts all input words$\}$.

B. $\{\langle \mathcal{M} \rangle \mid \mathcal{M}$ is a pushdown automaton that does not accept any input word$\}$.

C. $\{\langle \mathcal{M} \rangle \mid \mathcal{M}$ is a Turing machine that does not accept the empty word$\}$.

D. $\{\langle \mathcal{M}, \mathcal{N} \rangle \mid \mathcal{M}$ is a deterministic finite automaton that recognizes the same language as the nondeterministic finite automaton $\mathcal{N}\}$.

(j) If we know that X is a decision problem that is undecidable, and we also know that $X \leq_{\mathrm{m}} Y$ for some other decision problem Y, then what can we conclude about Y?

A. Y is decidable.

B. Y may or may not be decidable.

C. Y is undecidable.

D. Not enough information is given.

Short Answer

[10 marks] 2. For each of the following questions, give a 1–2 sentence answer.

(a) Why might we want a context-free grammar to be unambiguous?

(b) Why can't we use the pumping lemma for context-free languages to prove that a language is context-free?

(c) Suppose $k \geq 2$. How can we simulate the computation of a k-tape Turing machine using only a single tape?

(d) What does it mean for a decision problem to be undecidable?

(e) Describe how we proved that the decision problem A_{TM} is undecidable.
(You do not have to give a technical proof; a high-level description of the key idea is sufficient.)

Long Answer

[10 marks] 3. Let $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, and consider the language

$$L = \{\mathtt{a}^i \mathtt{b}^k \mathtt{c}^{2k} \mid i \geq 1, k \geq 0\}.$$

(a) Is this language L regular or context-free?

- If L is regular, prove this by constructing a finite automaton that recognizes L.
- If L is context-free, prove this by constructing a pushdown automaton that recognizes L.

Note. If L is regular, then you do not also have to prove it is context-free. If L is non-regular, then you do not need to prove non-regularity.

(b) Using your answer from part (a), give a trace of the behaviour of your automaton on the input word $w = \mathtt{aabbcccc}$. Your trace should list, at each step of the computation: the current state of the automaton, the current stack contents (if the language is context-free), and the remaining input word symbols.

[10 marks] 4. Construct a single-tape deterministic Turing machine $\mathcal{M}$ that decides the following language L over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

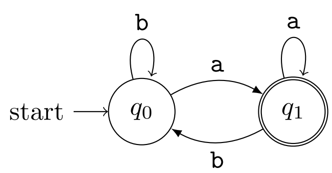
$$L = \{u\mathtt{c}v\mathtt{c}w \mid u, v, w \in \{\mathtt{a}, \mathtt{b}\}^{+}\}.$$

You should give each component of the tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$. You may optionally draw the Turing machine to illustrate your construction.

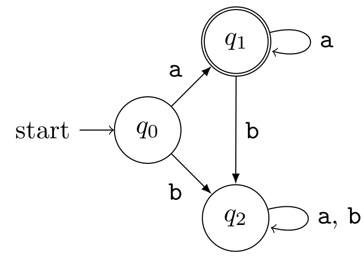
[12 marks] 5. (a) Consider the following decision problems:

$$A_{\mathsf{DFA}} = \{\langle \mathcal{B}, w\rangle \mid \mathcal{B} \text{ is a deterministic finite automaton that accepts input word } w\}$$
$$IN_{\mathsf{DFA}} = \{\langle \mathcal{B}, \mathcal{C}\rangle \mid \mathcal{B} \text{ and } \mathcal{C} \text{ are deterministic finite automata and } L(\mathcal{B}) \subseteq L(\mathcal{C})\}$$

Consider the following finite automata:



Finite automaton $\mathcal{M}_1$



Finite automaton $\mathcal{M}_2$

For each of the following questions, answer "yes" or "no" and justify your answer.

i. Is $\langle \mathcal{M}_1, \texttt{abba}\rangle \in A_{\mathsf{DFA}}$?

ii. Is $\langle \mathcal{M}_2, \texttt{aaba}\rangle \in A_{\mathsf{DFA}}$?

iii. Is $\langle \texttt{babbaaba}\rangle \in A_{\mathsf{DFA}}$?

iv. Is $\langle \mathcal{M}_1, \mathcal{M}_2\rangle \in IN_{\mathsf{DFA}}$?

(b) Consider the following decision problem:

$$STAR_{\mathsf{DFA}} = \{\langle \mathcal{B}\rangle \mid \mathcal{B} \text{ is a deterministic finite automaton and } L(\mathcal{B}) = a^* \text{ for some } a \in \Sigma\}.$$

Recall that Σ denotes the alphabet of the DFA $\mathcal{B}$. For example, if $\mathcal{B}$ were a DFA with the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$, then $\langle \mathcal{B}\rangle \in STAR_{\mathsf{DFA}}$ if either $L(\mathcal{B}) = \texttt{0}^*$ or $L(\mathcal{B}) = \texttt{1}^*$.
Prove that $STAR_{\mathsf{DFA}}$ is decidable by giving a decision algorithm.

[8 marks] 6. Consider the following decision problem:

$$B_{\mathsf{DFA,TM}} = \{\langle \mathcal{A}, \mathcal{M} \rangle \mid \mathcal{A} \text{ is a deterministic finite automaton, } \mathcal{M} \text{ is a Turing machine, and there exists an input word } w \text{ that is accepted by both } \mathcal{A} \text{ and } \mathcal{M}\}.$$

Prove that $B_{\mathsf{DFA,TM}}$ is undecidable by reducing from A_{TM}.

Hint. Remember that the directionality of your reduction is important: start from the assumption that $B_{\mathsf{DFA,TM}}$ is decidable and use this to construct a Turing machine for A_{TM}.

[2 marks] *Bonus.* What was your favourite part of this course, and why?

This blank page may be used for rough work.