Assignment Regulations.

- This assignment may be completed individually or in a group of two people. If you are collaborating on an assignment as a group, your group must submit exactly one joint set of answers.

- Please include your full name and email address on your submission. For groups, every member must include their full name and email address on the joint submission.

- You may either handwrite or typeset your submission. If your submission is handwritten, please ensure that the handwriting is neat and legible.

[10 marks] 1. (a) Let $P(x, y)$ denote the predicate "$x < y$" over the domain $\mathbb{R}$. For each of the following statements, express the statement as a predicate logic formula involving P.
   i. For every number, there is a smaller number.
   ii. For any two numbers, if the numbers are different, then one number is less than the other.
   iii. The number 7 is between a and b.

   (b) Recall the exclusive-or operation, $\oplus$. Give two predicates $P(x)$ and $Q(x)$ such that $\forall x\ (P(x) \oplus Q(x))$ is true, but $\forall x\ P(x) \oplus \forall x\ Q(x)$ is false. Specify the domain of your predicates. Explain why, given your predicates, the former formula is true while the latter formula is false.

[7 marks] 2. Suppose that we were allowed to use interpretations over an empty domain; that is, a domain D containing no elements.

   (a) Would a universally quantified formula $\forall x\ P(x)$ be true over an empty domain? Explain why or why not.

   (b) Would an existentially quantified formula $\exists x\ P(x)$ be true over an empty domain? Explain why or why not.

   (c) Using your observations from parts (a) and (b), what can be said about the equivalence

$$\forall y\ P(y, y) \vee \exists x\ Q(x, x) \equiv \exists x\ (\forall y\ P(y, y) \vee Q(x, x))$$

   if we take the domain to be empty? Does the equivalence hold or not hold?

[8 marks] 3. Using the method of semantic tableaux, determine whether the following formula is valid:

$$\forall x\ (P(x) \vee Q(x)) \Rightarrow (\forall x\ P(x) \vee \exists x\ Q(x)).$$

   For ease of writing, you can use the list format to prove validity instead of the tree format.

[15 marks] 4. Prove the validity of each of the following sequents using natural deduction.
   (a) $\forall x\ (P(x) \wedge Q(x)) \vdash \forall x\ P(x) \wedge \forall x\ Q(x)$.
   (b) $\forall x\ \neg P(x) \vdash \neg \exists x\ P(x)$.