CSCI 550: Advanced Algorithms
2. Extending Tractability

- *finding small vertex covers*
- *finding independent sets on trees*
- *circular arc coverings*
- *vertex cover in bipartite graphs*

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-complete problem. What should I do?
A. Theory says you're unlikely to find a poly-time algorithm.

Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve specific instances of the problem.

This lecture's focus. Solving some special cases of **NP**-complete problems.

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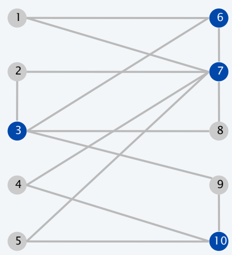
Vertex cover

Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$ or $v \in S$ or both?



S = { 3, 6, 7, 10 } is a vertex cover of size k = 4

Finding small vertex covers

VERTEX-COVER is **NP**-complete. But what if k is small?

Brute force. $O(k\, n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\, n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k; say, to $O(2^k\, k\, n)$.

Ex. $n = 1{,}000$, $k = 10$.
Brute. $k\, n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k\, k\, n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, then the algorithm is poly-time; if k is a *small* constant, then the algorithm is also practical.

Finding small vertex covers

Claim. Let (u, v) be an edge of G. G has a vertex cover of size $\leq k$ iff at least one of $G - \{ u \}$ and $G - \{ v \}$ has a vertex cover of size $\leq k - 1$.

delete v and all incident edges

Pf. [$\Rightarrow$]

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{ u \}$ is a vertex cover of $G - \{ u \}$.

Pf. [$\Leftarrow$]

- Suppose S is a vertex cover of $G - \{ u \}$ of size $\leq k - 1$.
- Then $S \cup \{ u \}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\leq k\,(n - 1)$ edges.

Pf. Each vertex covers at most $n - 1$ edges. ▪

Finding small vertex covers: algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\leq k$ in $O(2^k\, kn)$ time.

VERTEXCOVER (G, k)
IF (G contains no edges) RETURN true
IF (G contains $\geq kn$ edges) RETURN false

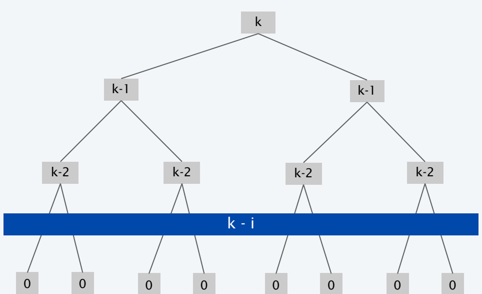
(u,v) = any edge of G
a = VERTEXCOVER $(G - \{u\}, k - 1)$
b = VERTEXCOVER $(G - \{v\}, k - 1)$
RETURN a or b

Pf.

- Correctness follows from the previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. ▪

Finding small vertex covers: recursion tree

$$T(n,k) \leq \begin{cases} c & \text{if } k = 0 \\ cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \leq 2^k c\, k\, n$$



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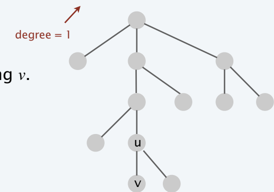
Independent set on trees

INDEPENDENT-SET is **NP**-complete for general graphs. But what about trees?

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree on at least two nodes has at least two leaf nodes.

Key observation. If v is a leaf, there exists a maximum size independent set containing v.



Pf. (exchange argument)

- Consider a max cardinality independent set S.
- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{ v \}$ is independent $\Rightarrow$ S not maximum.
- If $u \in S$ and $v \notin S$, then $S \cup \{ v \} - \{ u \}$ is independent. ▪

Independent set on trees: greedy algorithm

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
INDSETINFOREST (F)
  S = φ
  WHILE (F has at least one edge)
    (u,v) = any edge such that v is a leaf
    S = S ∪ v
    delete from F both u and v, and all edges incident to them
  RETURN S
```

Pf. Correctness follows from the previous key observation. ▪

Remark. Can implement in $O(n)$ time by considering nodes in postorder.

Weighted independent set on trees

Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S}\, w_v$.

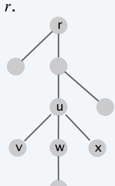
Observation. If (u, v) is an edge such that v is a leaf node, then either OPT includes u or OPT includes all leaf nodes incident to u.

Dynamic programming solution. Root tree at some node, say r.

- $OPT_{in}(u)$ = max weight independent set of subtree rooted at u, containing u.
- $OPT_{out}(u)$ = max weight independent set of subtree rooted at u, not containing u.

$$OPT_{in}(u) \;=\; w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) \;=\; \sum_{v \in \text{children}(u)} \max \{OPT_{in}(v),\ OPT_{out}(v)\}$$



children(u) = { v, w, x }

Weighted independent set on trees: dynamic programming algorithm

Theorem. The dynamic programming algorithm finds a maximum weighted independent set in a tree in $O(n)$ time.

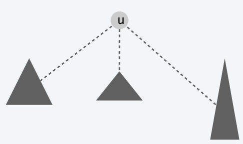
can also find independent set itself (not just value)

WEIGHTEDINDSETINTREE (T)
  root T at some node r
  FOREACH (node u of T in postorder)
    IF (u is a leaf)
      $M_{in}[u] = w_u$
      $M_{out}[u] = 0$
    ELSE
      $M_{in}[u] = w_u + \Sigma_{v \in \text{children}(u)} M_{out}[v]$
      $M_{out}[u] = \Sigma_{v \in \text{children}(u)} \max(M_{in}[v], M_{out}[v])$
  RETURN $\max(M_{in}[r], M_{out}[r])$

ensures a node is visited after all its children

Context

Independent set on trees. This structured special case is tractable because we can find a node that breaks the communication among the subproblems in different subtrees.



see Chapter 10.4 (but proceed with caution)

Graphs of bounded tree width. Elegant generalization of trees that:

- Captures a rich class of graphs that arise in practice.
- Enables decomposition into independent pieces.

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Wavelength-division multiplexing

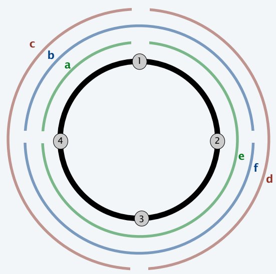
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case occurs when the network is a cycle on n nodes.

Bad news. **NP**-complete, even on rings.

Brute force. Can determine if k colours suffice in $O(k^m)$ time by trying all k-colourings.

Goal. $O(f(k)) \cdot poly(m, n)$ on rings.



n = 4, m = 6 **{ c, d }, { b, f }, { a, e }**

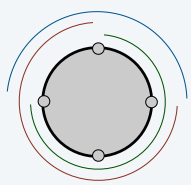
Review: interval colouring

Interval colouring. Greedy algorithm finds a colouring such that the number of colours equals the depth of the schedule.

maximum number of streams at one location

Circular arc colouring.

- Weak duality: number of colours $\geq$ depth.
- Strong duality does not hold.

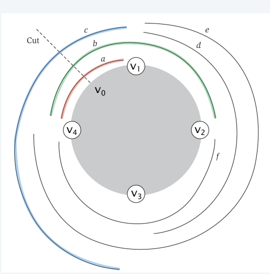


max depth = 2
min colours = 3

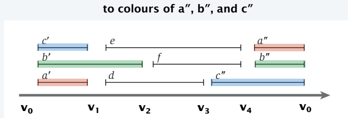
(Almost) transforming circular arc colouring to interval colouring

Circular arc colouring. Given a set of n arcs with depth $d \leq k$, can the arcs be coloured with k colours?

Equivalent problem. Cut the network between nodes v_1 and v_n. The arcs can be coloured with k colours iff the intervals can be coloured with k colours in such a way that "sliced" arcs have the same colour.



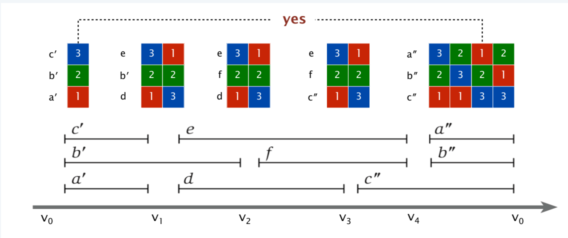
colours of a′, b′, and c′ must correspond to colours of a″, b″, and c″



Circular arc colouring: dynamic programming algorithm

Dynamic programming algorithm.

- Assign a distinct colour to each interval beginning at cut node v_0.
- At each node v_i, some intervals may finish, and others may begin.
- Enumerate all k-colourings of the intervals through v_i that are consistent with the colourings of the intervals through v_{i-1}.
- The arcs are k-colourable iff some colouring of intervals ending at cut node v_0 is consistent with the original colouring of the same intervals.



Circular arc colouring: running time

Running time. $O(k! \cdot n)$.

- The algorithm has n phases.
- Bottleneck in each phase: enumerating all consistent colourings.
- There are at most k intervals through v_i,
  so there are at most $k!$ colourings to consider.

Remark. This algorithm is practical for small values of k (say $k = 10$), even if the number of nodes n (or number of paths) is large.

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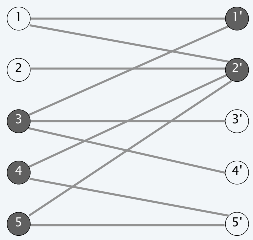
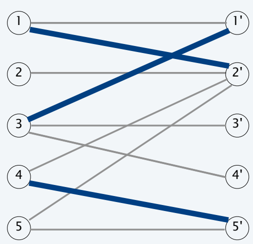


vertex cover S = { 3, 4, 5, 1', 2' }

Vertex cover and matching

Weak duality. Let M be a matching, and let S be a vertex cover. Then $|M| \leq |S|$.

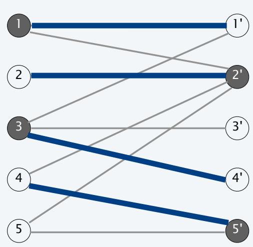
Pf. Each vertex can cover at most one edge in any matching. ▪



matching M: 1–2', 3–1', 4–5'

Vertex cover in bipartite graphs: König-Egerváry Theorem

Theorem. [König-Egerváry] In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

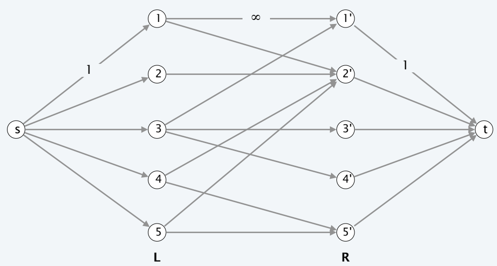


max matching M*: 1–1', 2–2', 3–4', 4–5'
min vertex cover S* = { 1, 3, 2', 5' }

Proof of König-Egerváry theorem

Theorem. [König-Egerváry] In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find a matching M and a cover S such that $|M| = |S|$.
- Formulate the max flow problem as for bipartite matching.
- Let M be a max cardinality matching and let (A, B) be a min cut.



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- Suffices to find a matching M and a cover S such that $|M| = |S|$.
- Formulate the max flow problem as for bipartite matching.
- Let M be a max cardinality matching and let (A, B) be a min cut.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$, $R_B = R \cap B$.

- Claim 1. $S = L_B \cup R_A$ is a vertex cover.
  - Consider $(u, v) \in E$.
  - $u \in L_A, v \in R_B$ is impossible because of infinite capacity.
  - Thus, either $u \in L_B$ or $v \in R_A$ or both.

- Claim 2. $|M| = |S|$.
  - Max-flow min-cut theorem $\Rightarrow |M| = cap(A, B)$.
  - Only edges of form (s, u) or (v, t) contribute to $cap(A, B)$.
  - $|M| = cap(A, B) = |L_B| + |R_A| = |S|$. ▪