University of Waterloo
CS240R Spring 2017
Sample Midterm Problems

Reminder: Midterm on Tuesday, June 20, 2017

Problem 1 (10 marks)

When Yu goes to sleep, his brain generates a number n and then runs an algorithm called $Dream(n, hour)$. This algorithm has two inputs: the number n and the hour at which Yu goes to sleep, which is either 12 AM, 1 AM, or 2 AM. The runtime of $Dream(n, hour)$ is n^i computations where i is the number of hours that Yu sleeps.

If Yu sleeps at 12 AM, he will get between 7 to 9 hours of sleep. If he goes to sleep at 1 AM, he will get exactly 6.5 hours of sleep. If he sleeps at 2 AM, he will get between 4 to 6 hours of sleep.

(a) True or False? In the worst case, the runtime of *Dream* is $\Theta(n^9)$. (2 marks)

(b) True or False? In the best case, the runtime of *Dream* is $\omega(n^4)$. (2 marks)

(c) True or False? If Yu sleeps at 1 AM, the runtime of *Dream* is $\Omega(n^{6.5})$. (2 marks)

(d) True or False? The runtime of *Dream* is $O(i^n)$. (2 marks)

(e) To study for the CS240 midterm, Yu changes his *Dream* algorithm so it runs in $\Theta(i)$ time where i is still the number of hours that he sleeps. True or False? The runtime of *Dream* is $o(1)$. (2 marks)

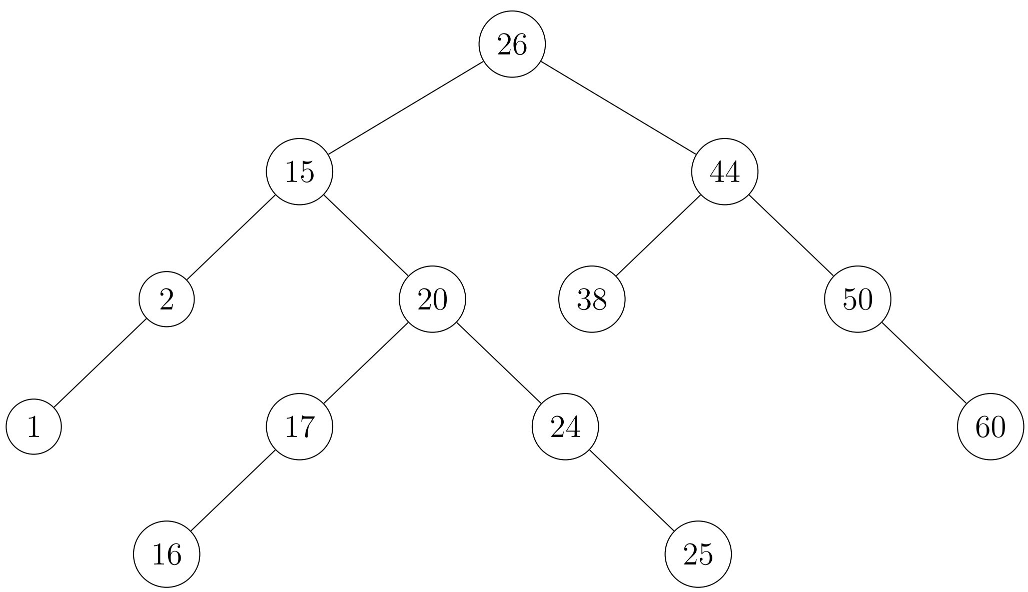
Problem 2 (8 marks)

Let $R_1, \ldots, R_n$ be n axis-aligned rectangles in the plane for which the corners are points in the $n \times n$-grid. Thus, for each rectangle R_i the four corners are points where both coordinates are integers in $\{1, \ldots, n\}$.

Give an algorithm to sort $R_1, \ldots, R_n$ by increasing area in $O(n)$ time.

Problem 3 (7 marks)

(a) An AVL tree is shown below without any balances. Write the balance at each node. (3 marks)



(b) In the above tree, show the result of calling delete(26) on the tree after all rotations are complete. (4 marks)

Problem 4 (9 marks)

Prove the following relations by first principles:

(a) $\arctan(\sqrt{\log n}) \in \Omega(\cos(n))$. (3 marks)

(b) $\sum_{i=1}^{\infty}(\frac{5}{8})^i \in \Theta(1)$. (3 marks)

(c) $\frac{1}{n^n} \in o(\frac{1}{n!})$. (3 marks)

Problem 5 (7 marks)

(a) Consider running Count Sort on the following numbers:

3	5	0	2	3	2	4	5	2	0	3

Draw the "left boundary" array that Count Sort creates initially for the numbers. (4 marks)

(b) Consider running LSD-radix sort on the following numbers:

27599	19473	52868	9238	42162	39587	69513

Write the position of each number after *one* iteration of LSD-radix sort. (3 marks)

Problem 6 (6 marks)

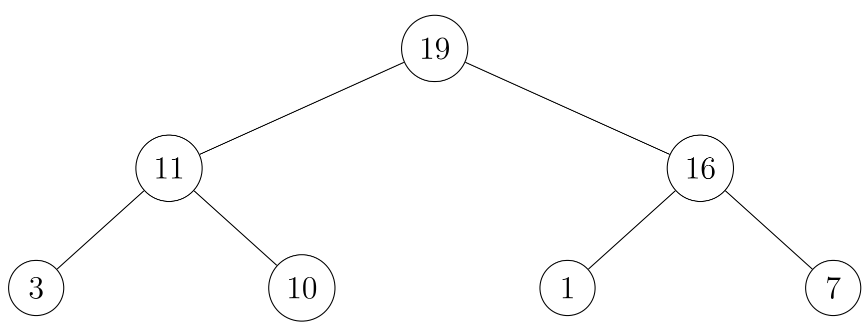
In a sorted array A of n elements from 1 to $n-1$, there is exactly one element that occurs twice. For example, in the following array of 10 elements, 6 occurs exactly twice:

A = 1, 2, 3, 4, 5, 6, 6, 7, 8, 9

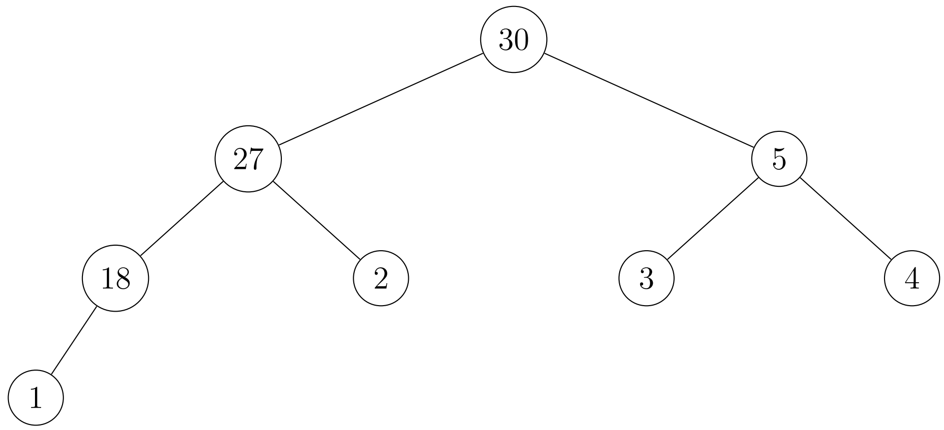
Create an $O(\log n)$ algorithm to find the repeating element in any such array.

Problem 7 (5 marks)

(a) Show the resulting max-heap after inserting 20. (2 marks)



(b) Show the resulting max-heap after calling $deleteMax()$. (3 marks)



Problem 8 (8 marks, 1 bonus)

Analyze the worst case runtime of the following algorithms.

(a) (4 marks)

```
s := n
p := 0
while (s >= 2)
    s = sqrt(s);
    p = p + 1;
```

(b) (4 marks)

```
s := n
while (s>0)
   if (s is even)
      s = floor(s/4)
   else
      s = 2*s
```

(c) (1 bonus mark)

```
!!!!!BONUS!!!!!

x := n
while (x > 1)
  if (x is even)
    x = x/2
  if (x is odd)
    x = 3*x + 1

!!!!!BONUS!!!!!
```

Total Marks: /60

Extra Problems

8c) not hard enough for you? Try some of these fun extra problems!

1. Prove by first principles that $fibonnaci(n) \in \omega(n^i)$ for any $i > 0$.
2. Consider the procedure Tortoise(int v) specified below.

```
Tortoise(int v)
   int ACHILLEAS := v
   real TORTOISE := 1
   int k:=1
   while (ACHILLEAS > TORTOISE) do
      ACHILLEAS:=ACHILLEAS+v
      TORTOISE:= TORTOISE*(k+1)/k
      TORTOISE:=TORTOISE+1
      k := k+1
```

a) Show that Tortoise(v) terminates for any positive integer v.

b) Argue that the running time of Tortoise(v) is $2^{\Theta(v)}$