

Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest priority*

deleteMax is also called *extractMax*.

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*.

Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

- *insert*: $O(1)$
- *deleteMax*: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use *sorted arrays*

- *insert*: $O(n)$
- *deleteMax*: $O(1)$

Using sorted linked-lists is identical.

Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:

A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc. .

Heaps

A *max-heap* is a binary tree with the following two properties:

- ① Structural Property: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
- ② Heap-order Property: For any node i , *key* (priority) of parent of i is larger than or equal to key of i .

A *min-heap* is the same, but with opposite order property.

Lemma: Height of a heap with n nodes is $\Theta(\log n)$.

Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *bubble-up*:

```
bubble-up( $v$ )
 $v$ : a node of the heap
1. while  $\text{parent}(v)$  exists and  $\text{key}(\text{parent}(v)) < \text{key}(v)$  do
2.     swap  $v$  and  $\text{parent}(v)$ 
3.      $v \leftarrow \text{parent}(v)$ 
```

The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$.

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *bubble-down*:

```
bubble-down( $v$ )
 $v$ : a node of the heap
1. while  $v$  is not a leaf do
2.      $u \leftarrow$  child of  $v$  with largest key
3.     if  $\text{key}(u) > \text{key}(v)$  then
4.         swap  $v$  and  $u$ 
5.          $v \leftarrow u$ 
6.     else
7.         break
```

Time: $O(\text{height of heap}) = O(\log n)$.

Priority Queue Realization Using Heaps

heapInsert(A, x)

A : an array-based heap, x : a new item

1. $size(A) \leftarrow size(A) + 1$
2. $A[size(A) - 1] \leftarrow x$
3. *bubble-up*($A, size(A) - 1$)

heapDeleteMax(A)

A : an array-based heap

1. $max \leftarrow A[0]$
2. *swap*($A[0], A[size(A) - 1]$)
3. $size(A) \leftarrow size(A) - 1$
4. *bubble-down*($A, 0$)
5. **return** max

Insert and deleteMax: $O(\log n)$

Storing Heaps in Arrays

Let H be a heap (binary tree) of n items and let A be an array of size n . Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i + 1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i + 2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).

Building Heaps

Problem statement: Given n items (in $A[0 \dots n - 1]$) build a heap containing all of them.

Solution 1: Start with an empty heap and insert items one at a time:

heapify1(A)

A : an array

1. initialize H as an empty heap
2. **for** $i \leftarrow 0$ **to** $size(A) - 1$ **do**
3. *heapInsert*($H, A[i]$)

This corresponds to going from $0 \dots n - 1$ in A and doing *bubble-ups*
Worst-case running time: $\Theta(n \log n)$.

Building Heaps

Problem statement: Given n items (in $A[0 \dots n - 1]$) build a heap containing all of them.

Solution 2: Using *bubble-downs* instead:

heapify(A)

A : an array

1. $n \leftarrow size(A) - 1$
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 0 **do**
3. *bubble-down*(A, i)

A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

Using a Priority Queue to Sort

PQ – *Sort*(*A*)

1. initialize *PQ* to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. *PQ.insert*(*A*[*i*], *A*[*i*])
4. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
5. $A[n - 1 - i] \leftarrow PQ.deleteMax()$

HeapSort

HeapSort(*A*)

1. initialize *H* to an empty heap
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. *heapInsert*(*H*, *A*[*i*])
4. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
5. $A[n - 1 - i] \leftarrow heapDeleteMax(H)$

HeapSort(*A*)

1. *heapify*(*A*)
2. **for** $i \leftarrow 0$ **to** $n - 1$ **do**
3. $A[n - 1 - i] \leftarrow heapDeleteMax(A)$

Running time of HeapSort: $O(n \log n)$

Selection

Problem Statement: The *k*th-max problem asks to find the *k*th largest item in an array *A* of *n* numbers.

Solution 1: Make *k* passes through the array, deleting the maximum number each time.

Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the *k*th largest number.

Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the *k* largest numbers seen so far in a min-heap

Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling *heapify*(*A*). Call *deleteMax*(*A*) *k* times.

Complexity: $\Theta(n + k \log n)$.