Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest priority*

deleteMax is also called *extractMax*.

Applications: typical "todo" list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*.

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- deleteMax: $O(1)$

Using sorted linked-lists is identical.

Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:
A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, *etc.* .

Heaps

A *max-heap* is a binary tree with the following two properties:

1. Structural Property: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
2. Heap-order Property: For any node i, *key* (priority) of parent of i is larger than or equal to key of i.

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Lemma: Height of a heap with n nodes is $\Theta(\log n)$.

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> *v: a node of the heap*
> 1. **while** *parent*(v) exists **and** *key*(*parent*(v)) $<$ *key*(v) **do**
> 2. swap v and *parent*(v)
> 3. $v \leftarrow$ *parent*(v)

The new item bubbles up until it reaches its correct place in the heap.

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Time: O(height of heap) $= O(\log n)$.

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
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```
bubble-down(v)
v: a node of the heap
1.    while v is not a leaf do
2.        u ← child of v with largest key
3.        if key(u) > key(v) then
4.            swap v and u
5.            v ← u
6.        else
7.            break
```

Time: O(height of heap) $= O(\log n)$.

Priority Queue Realization Using Heaps

heapInsert(A, x)
A: an array-based heap, x: a new item
1. $size(A) \leftarrow size(A) + 1$
2. $A[size(A) - 1] \leftarrow x$
3. $bubble-up(A, size(A) - 1)$

heapDeleteMax(A)
A: an array-based heap
1. $max \leftarrow A[0]$
2. $swap(A[0], A[size(A) - 1])$
3. $size(A) \leftarrow size(A) - 1$
4. $bubble-down(A, 0)$
5. **return** *max*

Insert and deleteMax: $O(\log n)$

Storing Heaps in Arrays

Let H be a heap (binary tree) of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i+1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i+2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).

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Problem statement: Given n items (in $A[0 \cdots n-1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
1.     initialize H as an empty heap
2.     for i ← 0 to size(A) − 1 do
3.         heapInsert(H, A[i])
```

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1. initialize H as an empty heap
2. **for** $i \leftarrow 0$ **to** *size*$(A) - 1$ **do**
3. *heapInsert*$(H, A[i])$

This corresponds to going from $0 \cdots n-1$ in A and doing *bubble-ups*
Worst-case running time: $\Theta(n \log n)$.

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Solution 2: Using *bubble-downs* instead:

heapify(A)
A: an array
1. $n \leftarrow size(A) - 1$
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 0 **do**
3. $\quad bubble-down(A, i)$

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A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

Using a Priority Queue to Sort

```
PQ − Sort(A)
    initialize PQ to an empty priority queue
    for i ← 0 to n − 1 do
        PQ.insert(A[i], A[i])
    for i ← 0 to n − 1 do
        A[n − 1 − i] ← PQ.deleteMax()
```

HeapSort

HeapSort(A)

initialize H to an empty heap
for $i \leftarrow 0$ **to** $n-1$ **do**
    heapInsert($H, A[i]$)
for $i \leftarrow 0$ **to** $n-1$ **do**
    $A[n-1-i] \leftarrow$ *heapDeleteMax*(H)

HeapSort

HeapSort(A)

```
initialize H to an empty heap
for i ← 0 to n − 1 do
    heapInsert(H, A[i])
for i ← 0 to n − 1 do
    A[n − 1 − i] ← heapDeleteMax(H)
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HeapSort(A)

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heapify(A)
for i ← 0 to n − 1 do
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```
HeapSort(A)
1.    initialize H to an empty heap
2.    for i ← 0 to n − 1 do
3.        heapInsert(H, A[i])
4.    for i ← 0 to n − 1 do
5.        A[n − 1 − i] ← heapDeleteMax(H)
```

```
HeapSort(A)
1.    heapify(A)
2.    for i ← 0 to n − 1 do
3.        A[n − 1 − i] ← heapDeleteMax(A)
```

Running time of HeapSort: $O(n \log n)$

Selection

Problem Statement: The kth-max problem asks to find the *kth largest item* in an array A of n numbers.

Solution 1: Make k passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the kth largest number.
Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the k largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling *heapify*(A). Call *deleteMax*(A) k times.
Complexity: $\Theta(n + k \log n)$.