

Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Dictionary ADT

A *dictionary* is a collection of *items*, each of which contains

- a *key*
- some *data*,

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- *search*(*k*)
- *insert*(*k*, *v*)
- *delete*(*k*)
- optional: *join*, *isEmpty*, *size*, etc.

Examples: symbol table, license plate database

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
(if not, the “value” could be a pointer)
- Comparing keys takes constant time

Unordered array or linked list

search $\Theta(n)$

insert $\Theta(1)$

delete $\Theta(n)$ (need to search)

Ordered array

search $\Theta(\log n)$

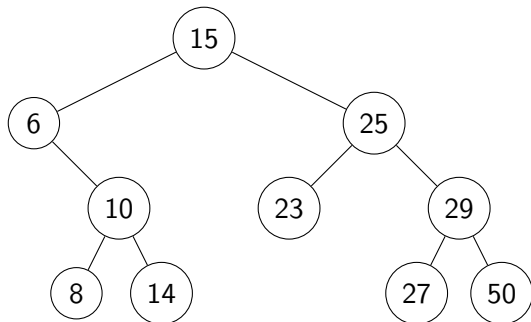
insert $\Theta(n)$

delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

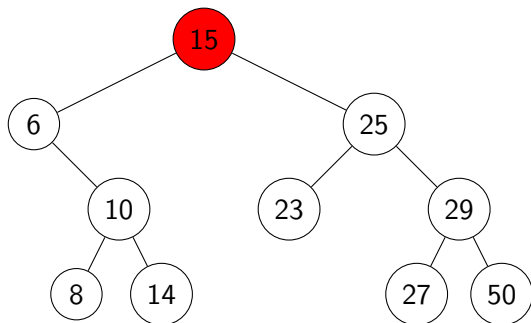
Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

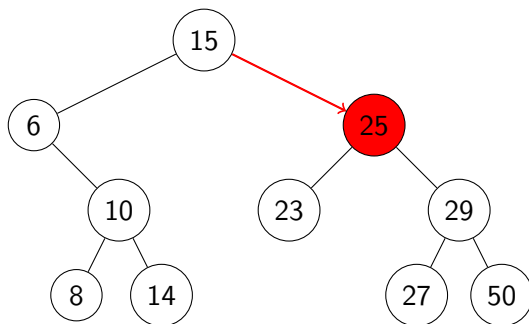
Example: *search*(24)



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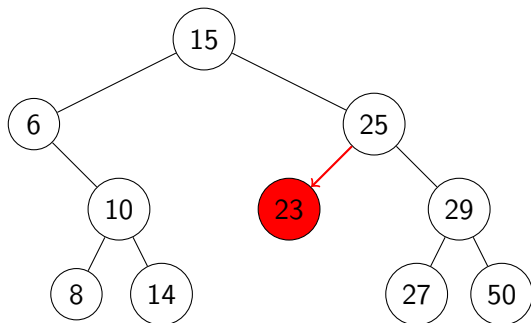
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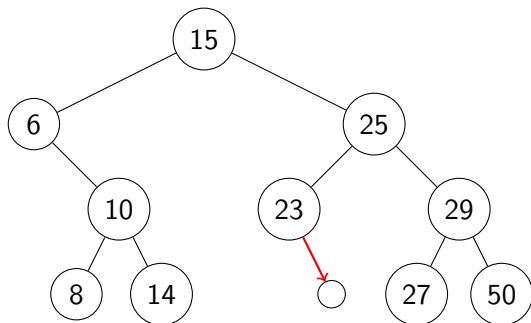
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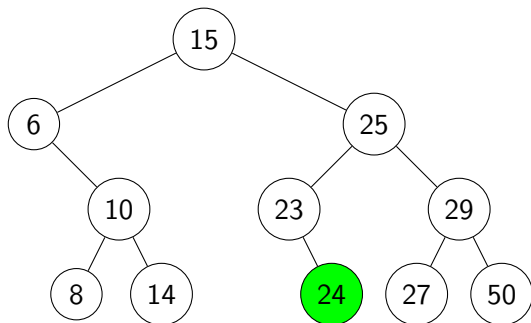


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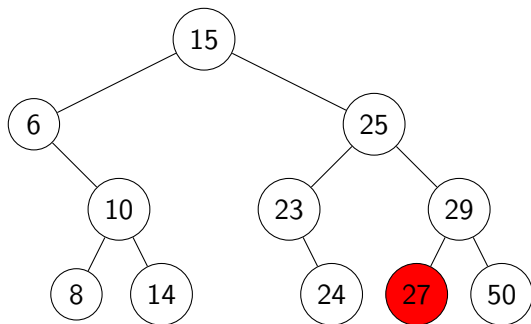
insert(k, v) Search for k , then insert (k, v) as new node

Example: *insert*(24, ...)



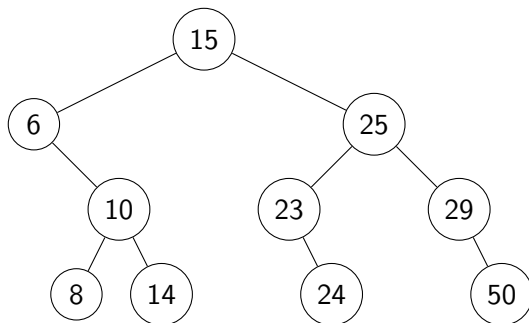
BST Delete

- If node is a leaf, just delete it.



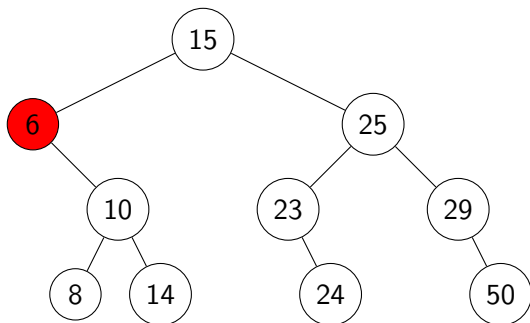
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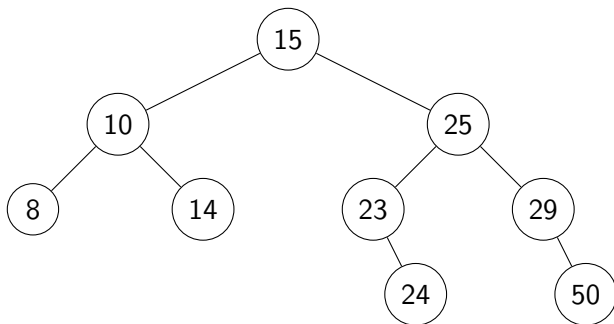
BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up



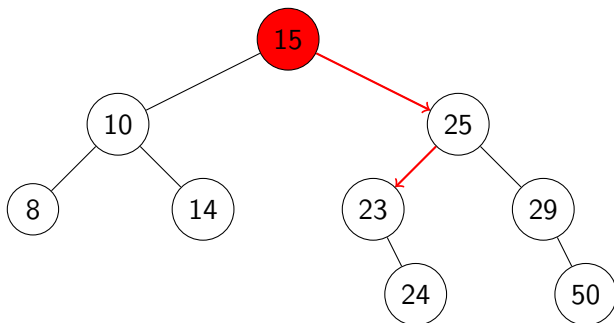
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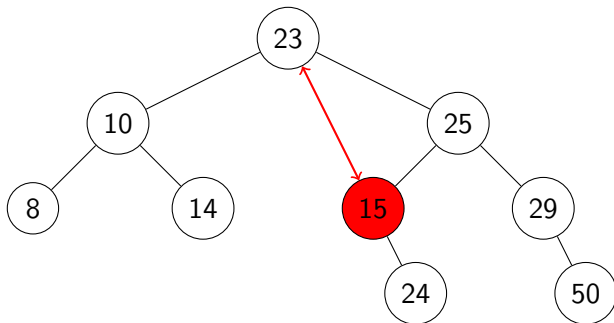
BST Delete

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- Else, swap with *successor* or *predecessor* node and then delete



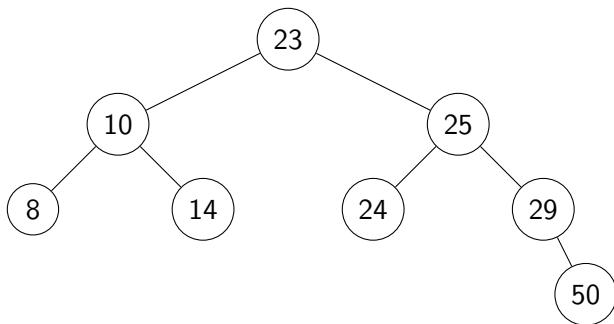
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Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
 h = height of the tree = max. path length from root to leaf

If n items are *inserted* one-at-a-time, how big is h ?

- Worst-case:

Height of a BST

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- Worst-case: $n - 1 = \Theta(n)$
- Best-case:

Height of a BST

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- Average-case:

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- Best-case: $\lfloor \lg(n) \rfloor = \Theta(\log n)$
- Average-case: $\Theta(\log n)$
(just like recursion depth in *quick-sort1*)

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962, an *AVL Tree* is a BST with an additional structural property:

The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1 .)

At each non-empty node, we store $height(R) - height(L) \in \{-1, 0, 1\}$:

-1 means the tree is *left-heavy*

0 means the tree is *balanced*

1 means the tree is *right-heavy*

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- We could store the actual height, but storing balances is simpler and more convenient.

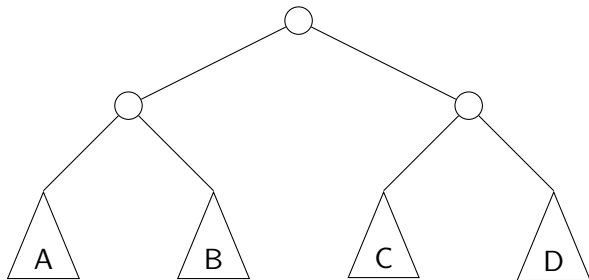
AVL insertion

To perform *insert*(T, k, v):

- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1 , 0 , or 1 , then keep going.
- If the balance factor is ± 2 , then call the *fix* algorithm to “rebalance” at that node. We are done.

How to “fix” an unbalanced AVL tree

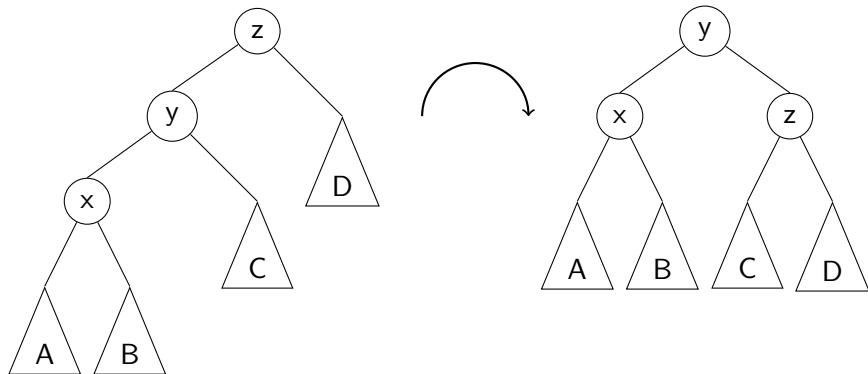
Goal: change the *structure* without changing the *order*



Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.

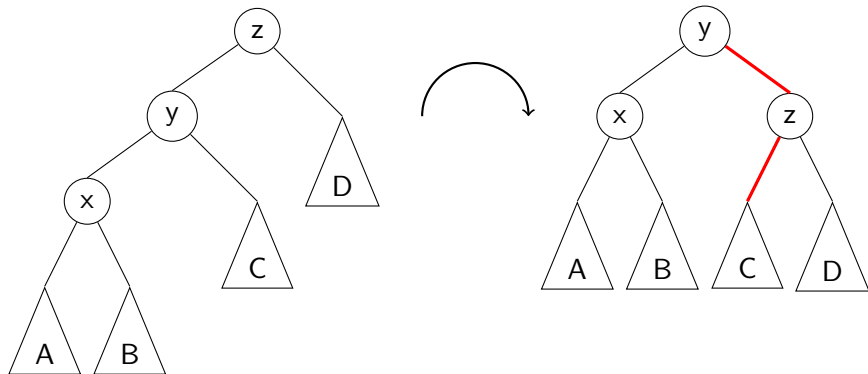
Right Rotation

This is a *right rotation* on node z:



Right Rotation

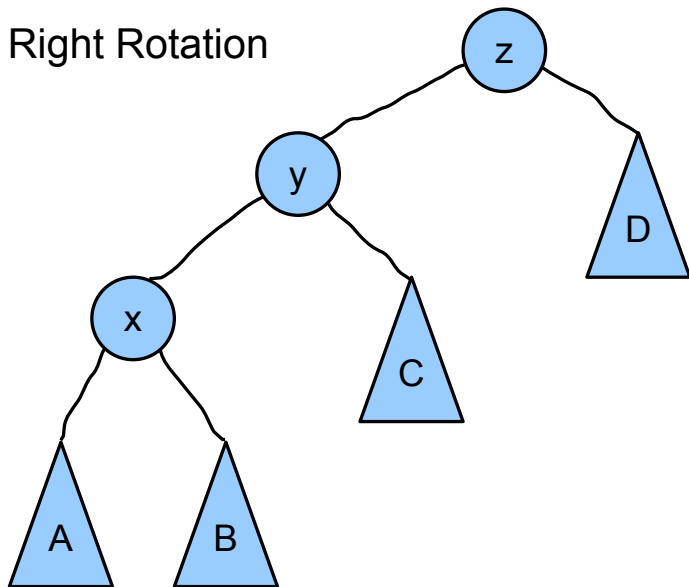
This is a *right rotation* on node z:



Note: Only two edges need to be moved, and two balances updated. Useful to fix left-left imbalance.

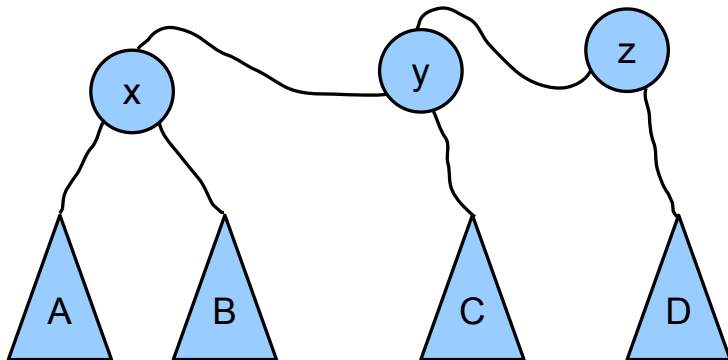
Again ...

Right Rotation



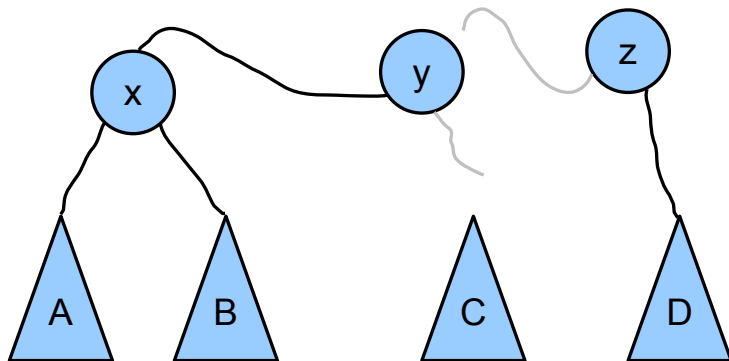
Again ...

Right Rotation



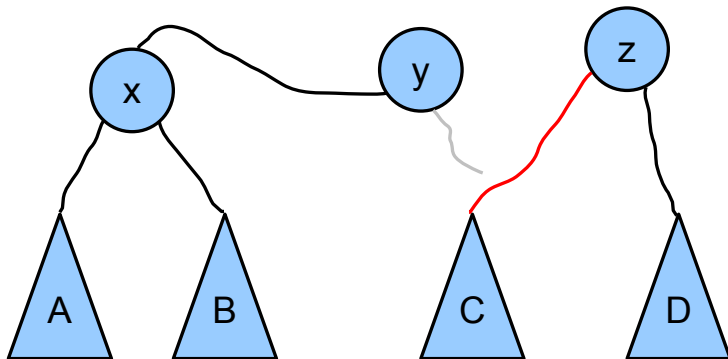
Again ...

Right Rotation



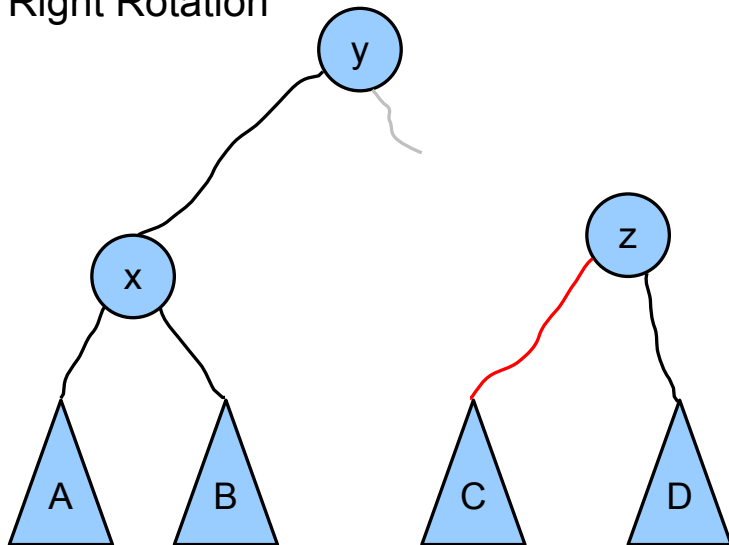
Again ...

Right Rotation



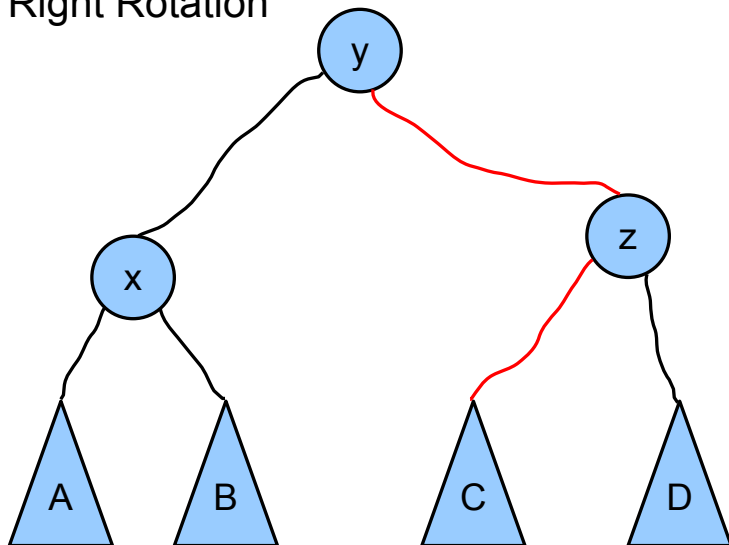
Again ...

Right Rotation



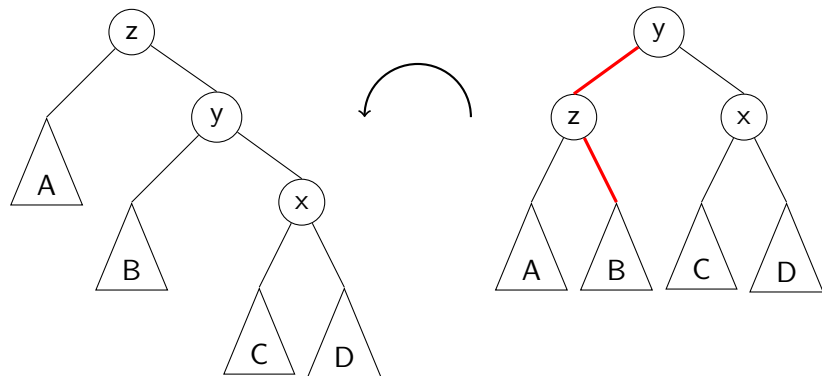
Again ...

Right Rotation



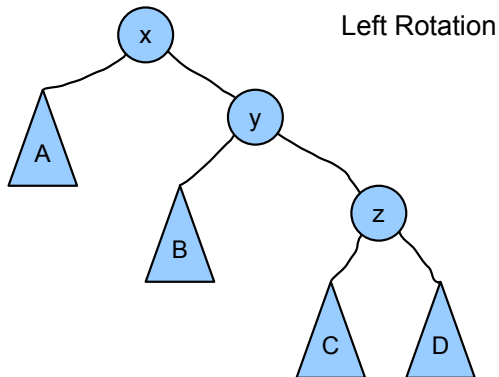
Left Rotation

This is a *left rotation* on node z:



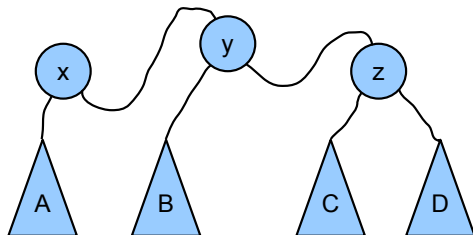
Again, only two edges need to be moved and two balances updated.
Useful to fix right-right imbalance.

Again ...



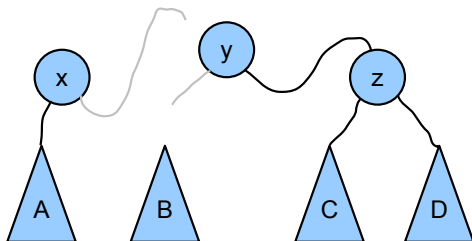
Again ...

Left Rotation



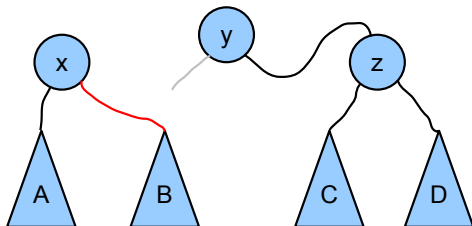
Again ...

Left Rotation

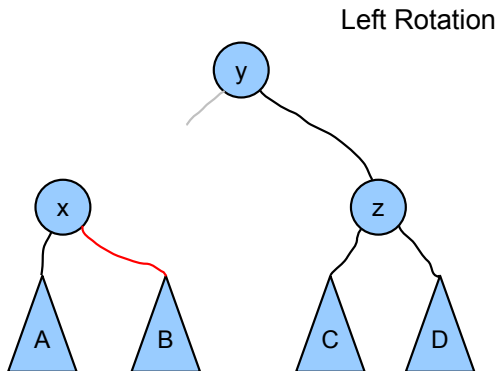


Again ...

Left Rotation

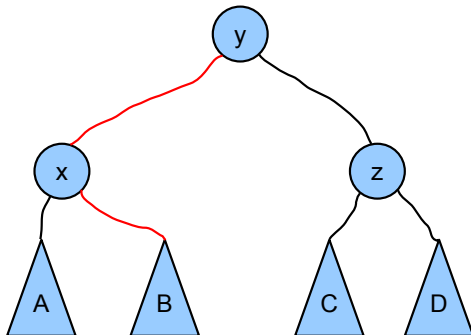


Again ...



Again ...

Left Rotation



Pseudocode for rotations

rotate-right(T)

T : AVL tree

returns rotated AVL tree

1. $newroot \leftarrow T.left$
2. $T.left \leftarrow newroot.right$
3. $newroot.right \leftarrow T$
4. **return** $newroot$

rotate-left(T)

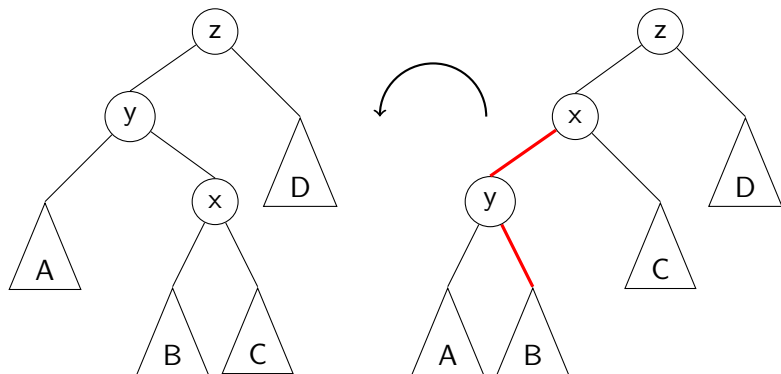
T : AVL tree

returns rotated AVL tree

1. $newroot \leftarrow T.right$
2. $T.right \leftarrow newroot.left$
3. $newroot.left \leftarrow T$
4. **return** $newroot$

Double Right Rotation

This is a *double right rotation* on node z :

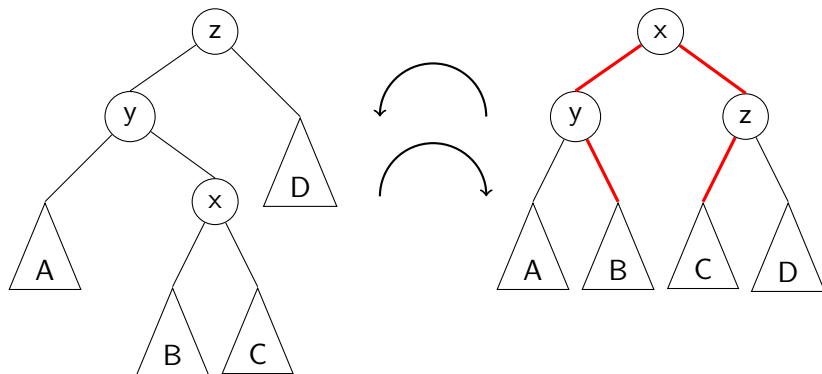


First, a left rotation on the left subtree (y). Second, a right rotation on the whole tree (z).

Useful for left-right imbalance.

Double Right Rotation

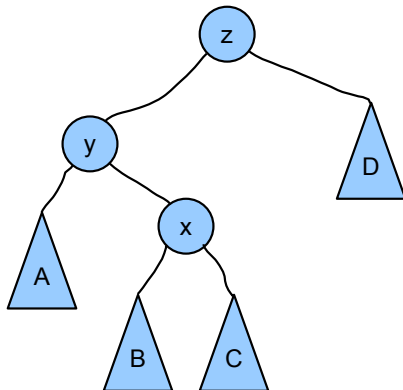
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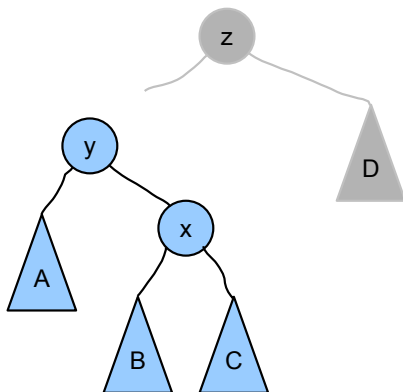
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Again ...



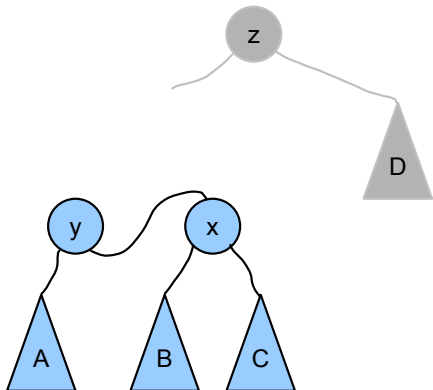
Double Right Rotation

Again ...



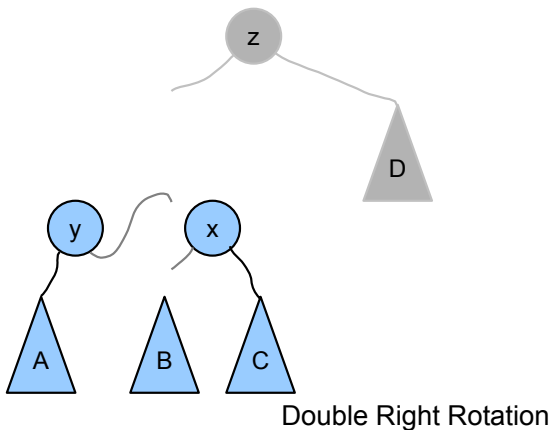
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Again ...

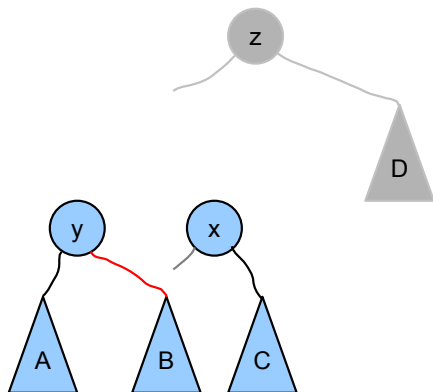


Double Right Rotation

Again ...

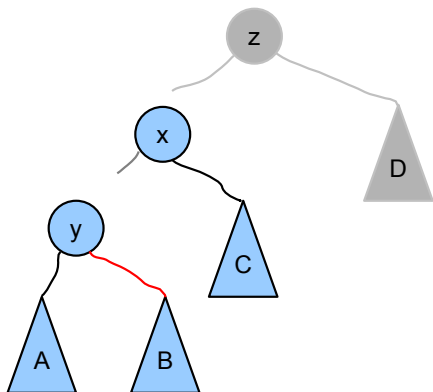


Again ...



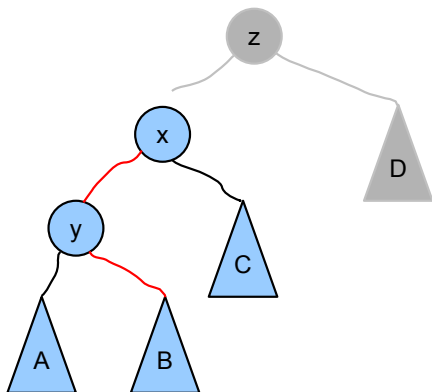
Double Right Rotation

Again ...



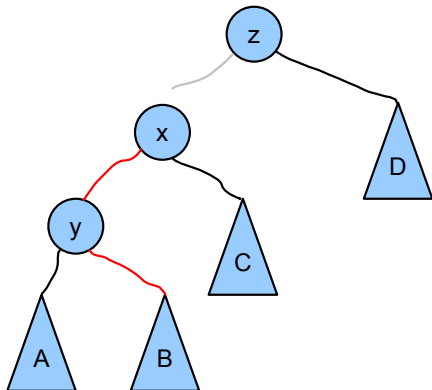
Double Right Rotation

Again ...



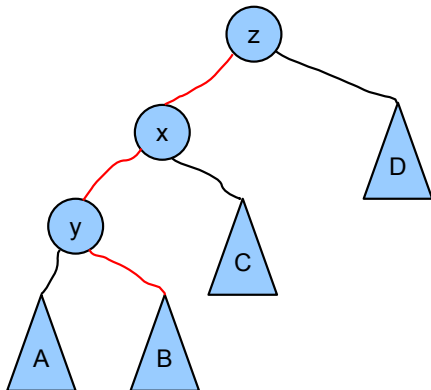
Double Right Rotation

Again ...



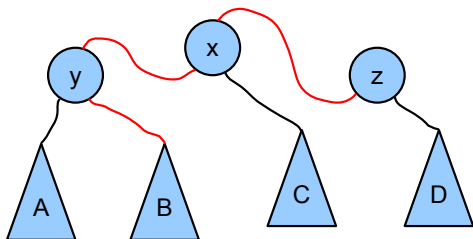
Double Right Rotation

Again ...



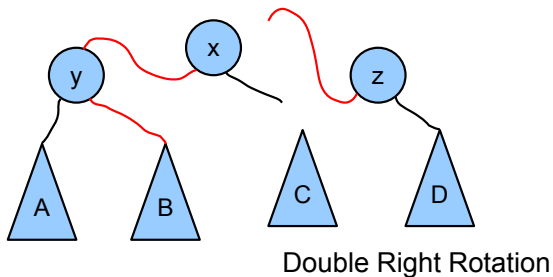
Double Right Rotation

Again ...

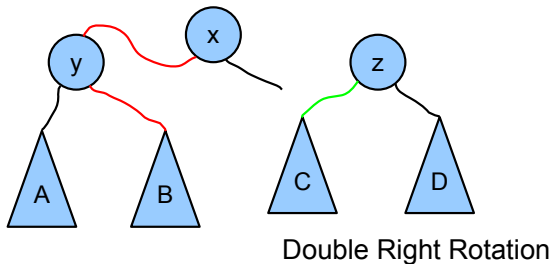


Double Right Rotation

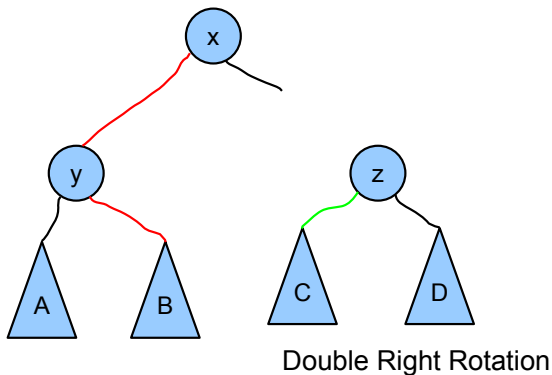
Again ...



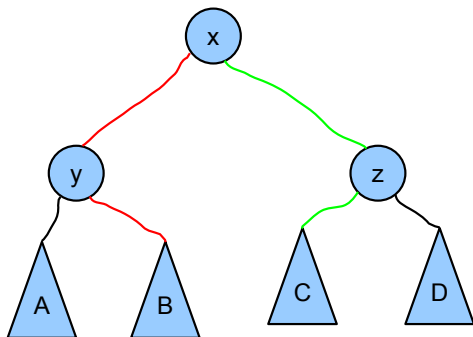
Again ...



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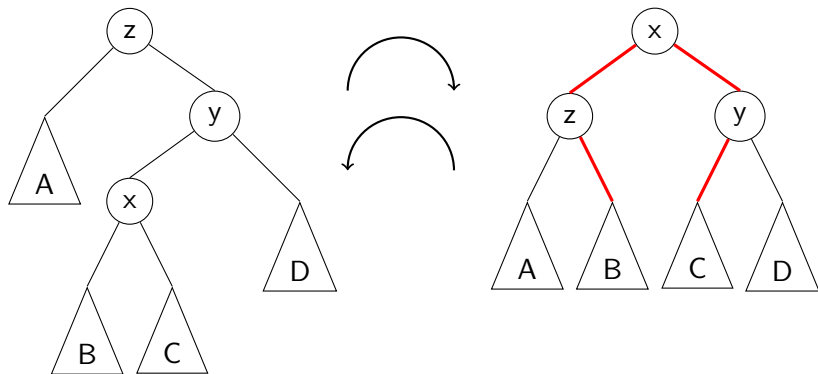
Again ...



Double Right Rotation

Double Left Rotation

This is a *double left rotation* on node z:



Right rotation on right subtree (y), followed by left rotation on the whole tree (z).

Useful for right-left imbalance.

Fixing a slightly-unbalanced AVL tree

Idea: Identify one of the previous 4 situations, apply rotations

fix(T)

T : AVL tree with $T.balance = \pm 2$

returns a balanced AVL tree

1. **if** $T.balance = -2$ **then**
2. **if** $T.left.balance = 1$ **then**
3. $T.left \leftarrow rotate_left(T.left)$
4. **return** $rotate_right(T)$
5. **else if** $T.balance = 2$ **then**
6. **if** $T.right.balance = -1$ **then**
7. $T.right \leftarrow rotate_right(T.right)$
8. **return** $rotate_left(T)$

AVL Tree Operations

search: Just like in BSTs, costs $\Theta(\text{height})$

insert: Shown already, total cost $\Theta(\text{height})$

- *fix* restores the height of the tree it fixes to what it was,
- so *fix* will be called *at most once*.

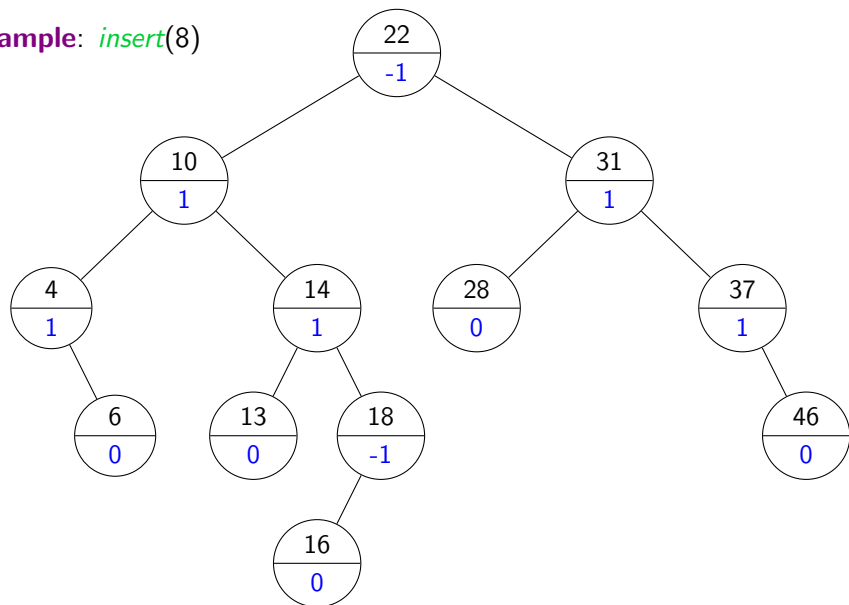
delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).

- *fix* may be called $\Theta(\text{height})$ times.

Total cost is $\Theta(\text{height})$.

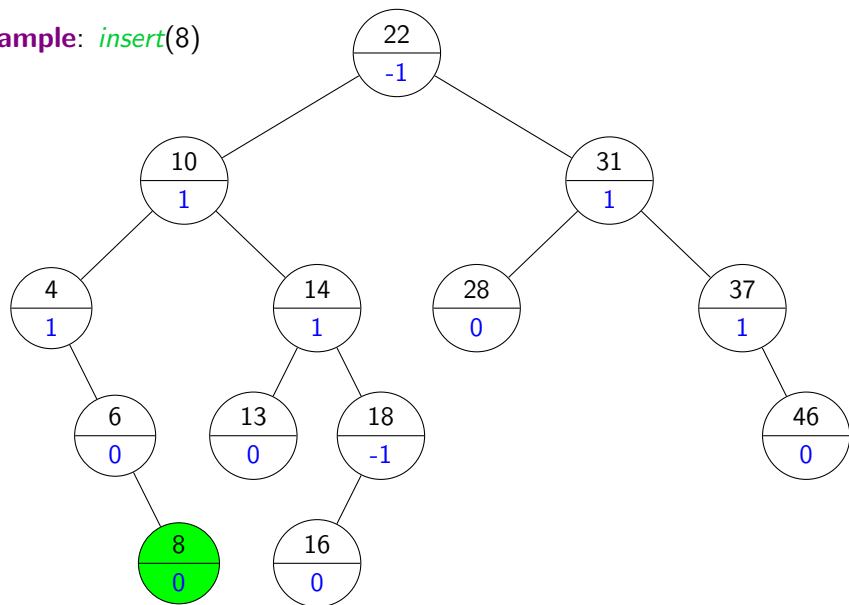
AVL tree examples

Example: *insert*(8)



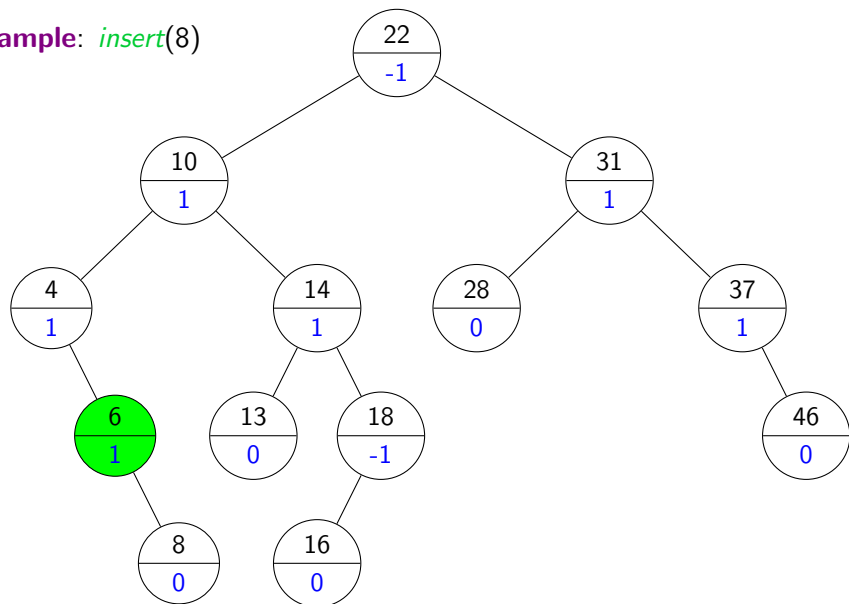
AVL tree examples

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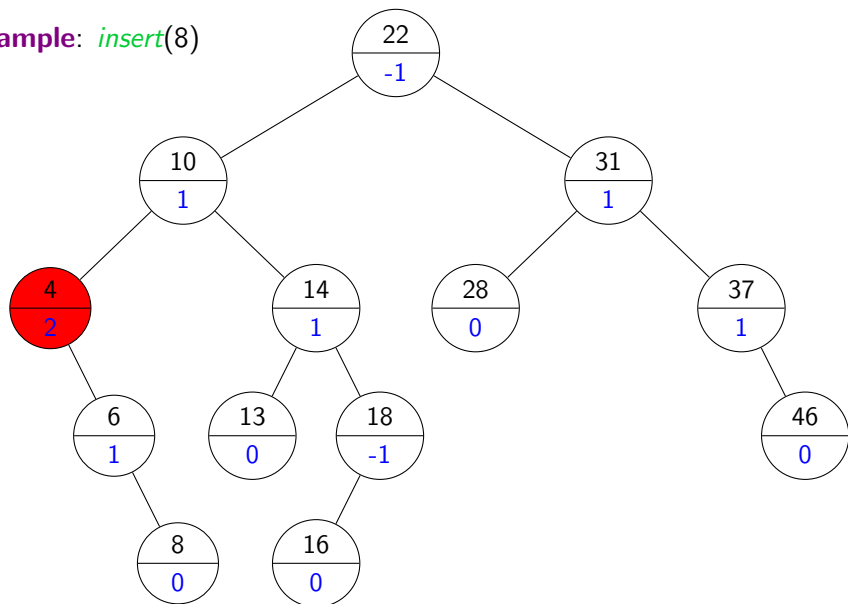
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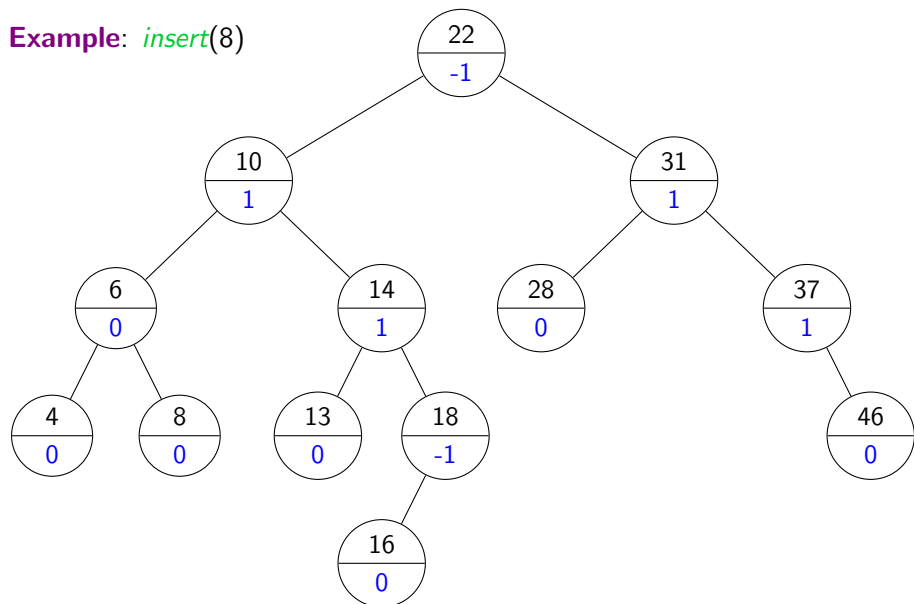
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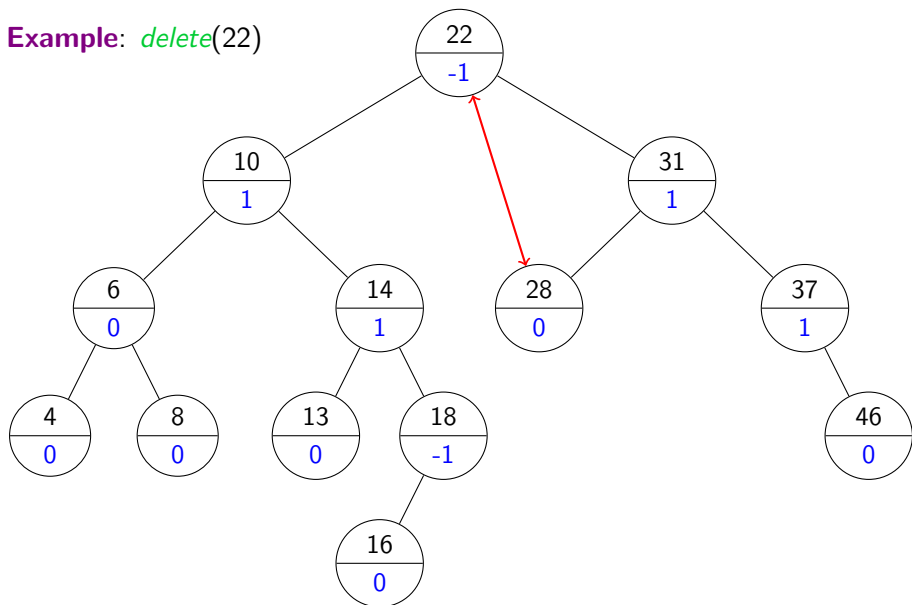
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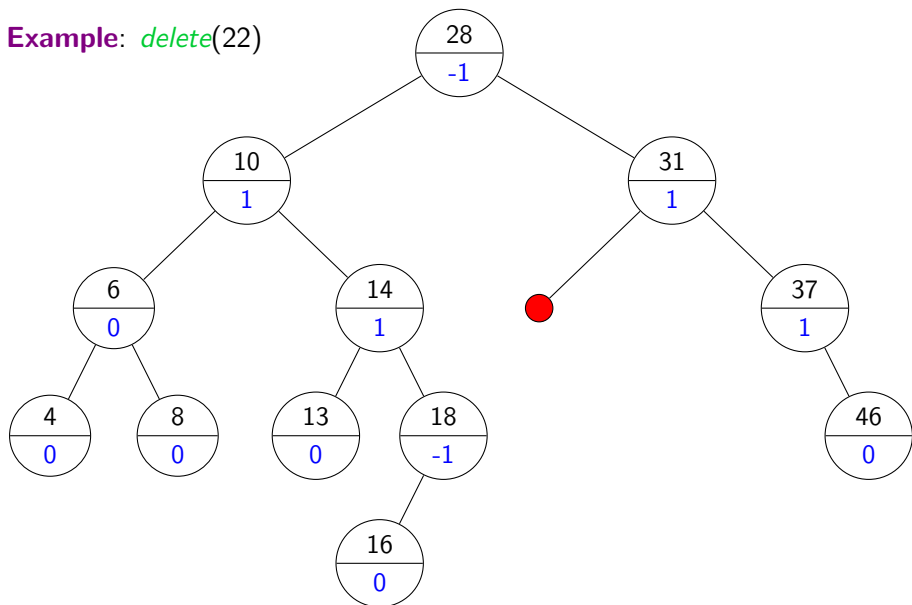
AVL tree examples

Example: *delete*(22)



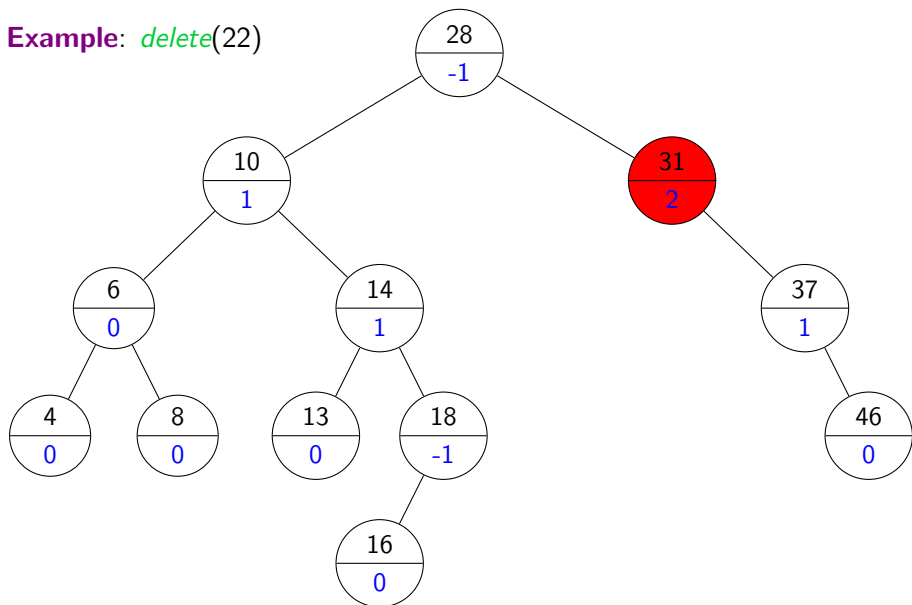
AVL tree examples

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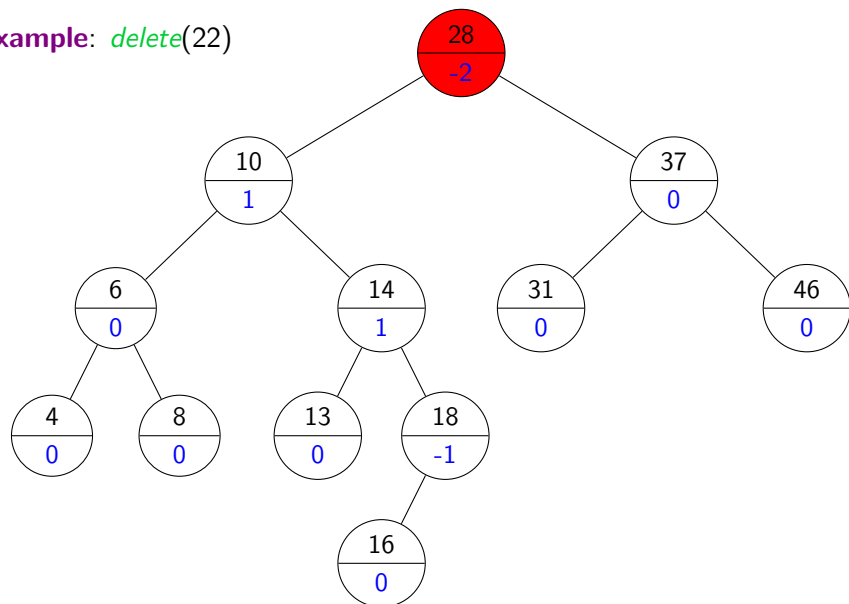
AVL tree examples

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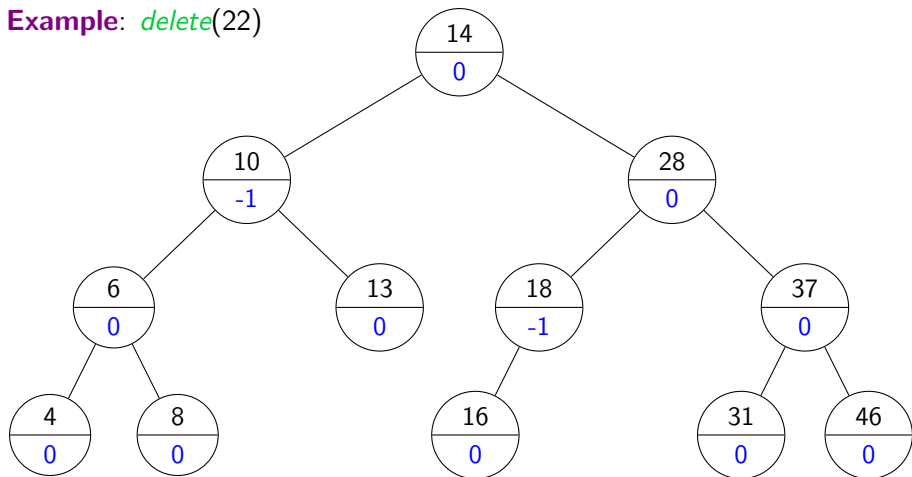
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AVL tree examples

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Height of an AVL tree

Define $N(h)$ to be the *least* number of nodes in a height- h AVL tree.

One subtree must have height at least $h - 1$, the other at least $h - 2$:

$$N(h) = \begin{cases} 1 + N(h - 1) + N(h - 2), & h \geq 1 \\ 1, & h = 0 \\ 0, & h = -1 \end{cases}$$

What sequence does this look like?

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What sequence does this look like? The Fibonacci sequence!

$$N(h) = F_{h+3} - 1 = \left\lceil \frac{\varphi^{h+3}}{\sqrt{5}} \right\rceil - 1, \text{ where } \varphi = \frac{1 + \sqrt{5}}{2}$$

AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \dots > 2^i N(h-2i) \geq 2^{\lfloor h/2 \rfloor}$$

AVL Tree Analysis

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Since $n > 2^{\lfloor h/2 \rfloor}$, $h \leq 2 \lg n$,

and thus an AVL tree with n nodes has height $O(\log n)$.

Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

\Rightarrow *search*, *insert*, *delete* all cost $\Theta(\log n)$.