

# Module 5: Dictionaries II

CS 240 - Data Structures and Data Management

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## Dictionary ADT: Review

A *dictionary* is a collection of *key-value pairs* (KVPs), supporting operations *search*, *insert*, and *delete*.

### Realizations

- **Unordered array or linked list:**  $\Theta(1)$  insert,  $\Theta(n)$  search and delete
- **Ordered array:**  $\Theta(\log n)$  search,  $\Theta(n)$  insert and delete
- **Balanced search trees (AVL trees):**  
 $\Theta(\log n)$  search, insert, and delete

## Self-Organizing Search

- Unordered linked list  
*search*:  $\Theta(n)$ , *insert*:  $\Theta(1)$ , *delete*:  $\Theta(1)$  (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?
- No: if items are accessed equally likely
- Yes: otherwise (we have a probability distribution for items)
- **Optimal static ordering**: sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.
- **Proof Idea**: For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

## Optimal Static Ordering

**A list of elements ordered by non-increasing probability of access has minimum expected access cost**

- $L = \langle x_1, x_2, \dots, x_n \rangle$   
Expected access cost in  $L$  is  
$$E(L) = \sum_{i=1}^n P(x_i) T(x_i) = \sum_{i=1}^n P(x_i) \cdot i$$
$$P(x_i)$$
 - access probability for  $x_i$   
$$T(x_i)$$
 - position of  $x_i$  in  $L$
- **Example**  
$$P(a) = 0.3 \quad P(b) = 0.5 \quad P(c) = 0.2$$
$$L = \langle a, b, c \rangle$$
$$E(L) = 0.3 + 0.5 * 2 + 0.2 * 3 = 1.9$$
$$L = \langle b, a, c \rangle$$
$$E(L) = 0.5 + 0.3 * 2 + 0.2 * 3 = 1.7$$

## Optimal Static Ordering

**A list of elements ordered by non-increasing probability of access has minimum expected access cost**

Proof by Contradiction

- $L = \langle x_1, \dots, x_k, x_{k+1}, \dots, x_n \rangle$

Suppose the access cost of  $L$  is optimal and there is  $k$  such that  $P(x_k) < P(x_{k+1})$

$$E(L) = P(x_k) \cdot k + P(x_{k+1}) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i$$

- Create another list  $L'$  by swapping  $x_k$  and  $x_{k+1}$ .

$$L' = \langle x_1, \dots, x_{k+1}, x_k, \dots, x_n \rangle$$

$$E(L') = P(x_{k+1}) \cdot k + P(x_k) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i$$

- $E(L') - E(L) = P(x_k) - P(x_{k+1}) < 0 \Rightarrow E(L') < E(L)$

Contradiction

## Dynamic Ordering

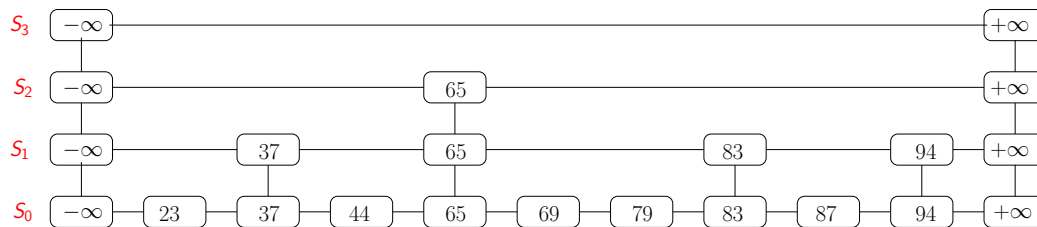
- What if we do not know the access probabilities ahead of time?
- **Move-To-Front(MTF)**: Upon a successful search, move the accessed item to the front of the list
- **Transpose**: Upon a successful search, swap the accessed item with the item immediately preceding it

**Performance of dynamic ordering:**

- Both can be implemented in arrays or linked lists.
- Transpose does not adapt quickly to changing access patterns.
- MTF Works well in practice.
- Theoretically MTF is “competitive”:  
No more than twice as bad as the optimal “offline” ordering.

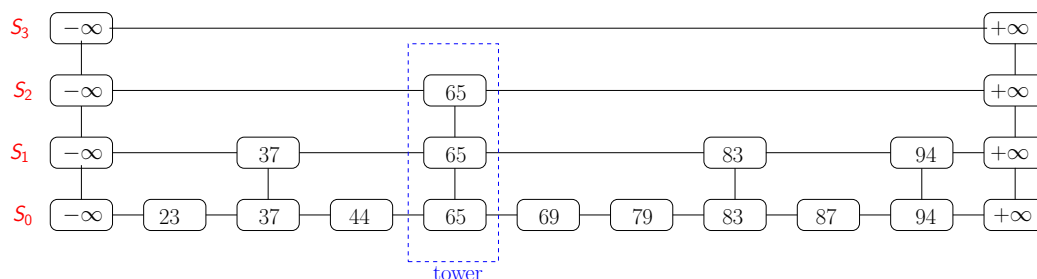
# Skip Lists

- **Randomized** data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A **skip list** for a set  $S$  of items is a series of lists  $S_0, S_1, \dots, S_h$  such that:
  - ▶ Each list  $S_i$  contains the special keys  $-\infty$  and  $+\infty$
  - ▶ List  $S_0$  contains the keys of  $S$  in non-decreasing order
  - ▶ Each list is a subsequence of the previous one, i.e.,  $S_0 \supseteq S_1 \supseteq \dots \supseteq S_h$
  - ▶ List  $S_h$  contains only the two special keys



# Skip Lists

- A **skip list** for a set  $S$  of items is a series of lists  $S_0, S_1, \dots, S_h$
- A two-dimensional collection of positions: **levels** and **towers**
- Traversing the skip list:  $\text{after}(p)$ ,  $\text{below}(p)$



# Search in Skip Lists

*skip-search*( $L, k$ )

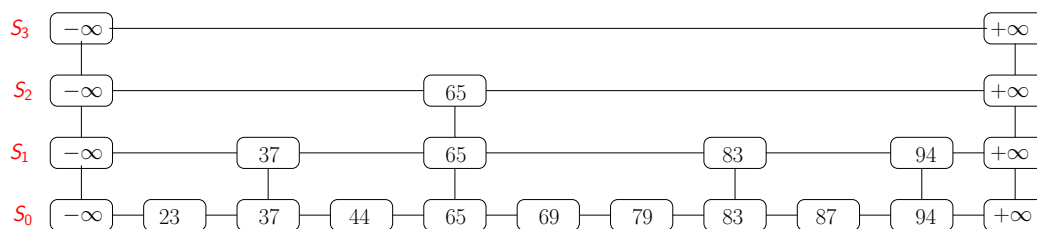
$L$ : A skip list,  $k$ : a key

1.  $p \leftarrow$  topmost left position of  $L$
2.  $S \leftarrow$  stack of positions, initially containing  $p$
3. **while**  $below(p) \neq null$  **do**
4.      $p \leftarrow below(p)$
5.     **while**  $key(after(p)) < k$  **do**
6.          $p \leftarrow after(p)$
7.     push  $p$  onto  $S$
8. **return**  $S$

- $S$  contains positions of the largest key **less than**  $k$  at each level.
- $after(top(S))$  will have key  $k$ , iff  $k$  is in  $L$ .
- **drop down:**  $p \leftarrow below(p)$
- **scan forward:**  $p \leftarrow after(p)$

# Search in Skip Lists

Example: Skip-Search( $S, 87$ )

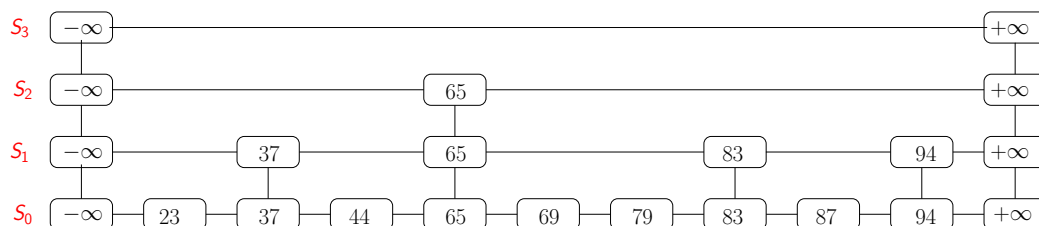


## Insert in Skip Lists

- $Skip\text{-}Insert(S, k, v)$ 
  - ▶ Randomly compute the height of new item: repeatedly toss a coin until you get tails, let  $i$  be the number of times the coin came up heads
  - ▶ Search for  $k$  in the skip list and find the positions  $p_0, p_1, \dots, p_i$  of the items with largest key less than  $k$  in each list  $S_0, S_1, \dots, S_i$  (by performing  $Skip\text{-}Search(S, k)$ )
  - ▶ Insert item  $(k, v)$  into list  $S_j$  after position  $p_j$  for  $0 \leq j \leq i$  (a tower of height  $i$ )

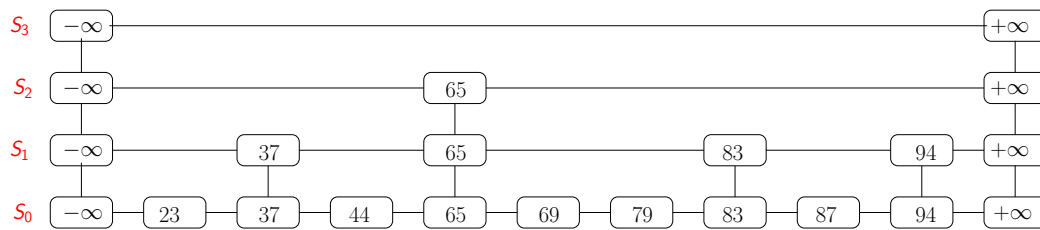
## Insert in Skip Lists

Example:  $Skip\text{-}Insert(S, 52, v)$



## Insert in Skip Lists

Example: Skip-Insert( $S, 100, v$ )

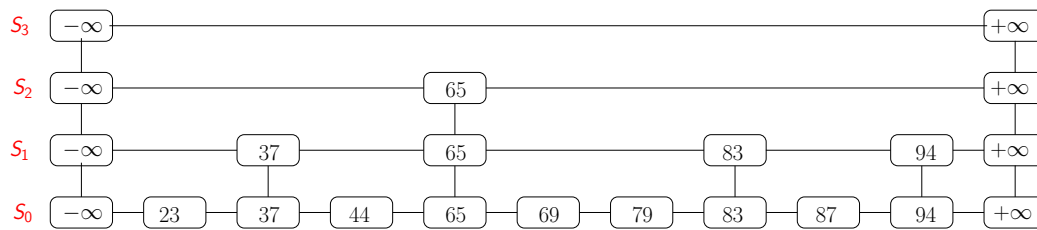


## Delete in Skip Lists

- Skip-Delete( $S, k$ )
  - ▶ Search for  $k$  in the skip list and find all the positions  $p_0, p_1, \dots, p_i$  of the items with the largest key smaller than  $k$ , where  $p_j$  is in list  $S_j$ . (this is the same as Skip-Search)
  - ▶ For each  $i$ , if  $\text{key}(\text{after}(p_i)) == k$ , then remove  $\text{after}(p_i)$  from list  $S_i$
  - ▶ Remove all but one of the lists  $S_i$  that contain only the two special keys

## Delete in Skip Lists

Example: Skip-Delete( $S, 65$ )



## Summary of Skip Lists

- Expected **space** usage:  $O(n)$
- Expected **height**:  $O(\log n)$   
A skip list with  $n$  items has height at most  $3 \log n$  with probability at least  $1 - 1/n^2$
- *Skip-Search*:  $O(\log n)$  expected time
- *Skip-Insert*:  $O(\log n)$  expected time
- *Skip-Delete*:  $O(\log n)$  expected time
- Skip lists are fast and simple to implement in practice